(12)

NSWC/DL TR-3624

ADA037381

MAP PROJECTION EQUATIONS

by

FREDERICK PEARSON II

Warfare Analysis Department

MARCH 1977

Approved for public release; distribution unlimited.

NAVAL SURFACE WEAPONS CENTER
Dahlgren Laboratory
Dahlgren, Virginia 22448

nswc

NAVAL SURFACE WEAPONS CENTER
DAHLGREN LABORATORY
Dahlgren, Virginia
22448

D. M. Agnew, Jr., Capt., USN
OIC and Assistant Commander

DISCLAIMER NOTICE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: TR-3624	2. GOVT ACCESSION NO.
	3. RECIPIENT'S CATALOG NUMBER
6 4. TITLE (and Subtitle): MAP PROJECTION EQUATIONS	9 5. TYPE OF REPORT & PERIOD COVERED: Final rept.,
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): 10 Frederick Pearson, II	8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Naval Surface Weapons Center (DK-13) Dahlgren Laboratory Dahlgren, Virginia 2^448	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 12 339p.
11. CONTROLLING OFFICE NAME AND ADDRESS: Defense Mapping Agency Washington, DC 236	11 12. REPORT DATE: March 1977
	13. NUMBER OF PAGES: 342
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): 14 NSWC/DL-TR-3624	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

DDC

MAR 29 1977

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Map Projections
Conformal Mapping
Equal Area Mapping
Conventional Mapping
Differential Geometry
Figure of the Earth
Distortion in Maps

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Map projection equations in cartesian coordinates are derived for the most important mapping schemes. The important unifying principles of differential geometry are applied to produce the equal area, conformal and conventional projections. This report has collected under one cover the major map projections useful to scientists and engineers. The notation is uniform. The derivations proceed in all cases from first principles to usable equations suitable for hand plotting, digital/analog plotting, or CRT display.

(see back)

391 598

(20)

The problem is stated, and the reader is introduced to the terminology of the art of map projections. Basic transformation theory is introduced, and then particularized for the transformation from the spheroid or sphere onto a developable surface. The criterion for the derivations is to use the most simple and direct approach.

The model of the earth is then considered. The most recent parameters to describe the figure of the earth are given, and tables incorporating these are included for meridian length, parallel length and the relation between geodetic and geocentric latitude. The computer programs which generated these tables are included in the appendix.

Equal area, conformal, and conventional projection equations are derived. These equations are incorporated into an original computer program which generated the map plotting tables for the most important projections. This program, which produces either a complete grid or individualized points, is also in the appendix. Since the proof of all of the derivations is a correct graticule of meridians and parallels, original figures of these have been produced. The plotting tables and the figures reflect the modern parameters for the earth.

The criterion for the success of any projection scheme, and the tool for selecting the most useful scheme for and application is obtained by considering the theory of distortions. A numerical method is introduced which permits a quantitative estimate of linear and angular distortion.

FOREWORD

This report derives the mapping equations for the majority of map projections in current use, based on the unifying principles of differential geometry. The publication of this report was sponsored by the Defense Mapping Agency, Washington, D.C.

This report has been reviewed by Mr. R. J. Anderle, Head, Astronautics and Geodesy Division and Dr. Leonard Merrovitch, ESM Advisor, VPI & SU.

Released by:

for R. A. NIEMANN, Head
Warfare Analysis Department

ACKNOWLEDGEMENTS

My thanks go to Mr. Richard J. Anderle, NSWC, and Dr. Alvin V. Hershey, NSWC, for review and criticism, and to Mr. Raymond B. Manrique, NSWC, and Mr. Kenneth Nelson, DMA, for making information available.

My thanks and the dedication go to my wife Betty.

TABLE OF CONTENTS

Chapter		Page

Chapter			Page

FIGURES

TABLES

Chapter 1

INTRODUCTION

Map projection is the orderly transfer of positions of places on the surface of the earth to corresponding points on a flat sheet of paper, a map. The process of transformation requires a degree of approximation and simplification. This first chapter lays the ground-work for the study by detailing, in a qualitative way, the basic problem and introducing the nomenclature of maps. Succeeding chapters will consider the mathematical techniques and the simplifications required to obtain manageable solutions [9].*

All projections introduce distortions in the map. The types of distortion are considered in terms of length, angle, and area. This chapter discusses the qualitative aspects of the problem, while Chapter 7 deals with it quantitatively.

The coordinate systems useful in locating positions on the earth, and on the map are summarized. The concept of scale factor to reduce earth sized lengths to map sized lengths is discussed.

Map projections may be classified in a number of ways. The principle one is by the features preserved from distortion by the mapping technique. Other methods of classification depend on the plotting surface employed, the method of contact of this surface with the earth, and the orientation of the plotting surface with respect to the direction of the earth's polar axis. Finally, maps can be classified according to whether or not a map can be drawn by purely graphical means.

The convention for azimuth used in this volume is also introduced.

1.1 Introduction to the Problem

Map projection requires the transformation of positions from a curved surface, the earth, onto a plane surface, the map, in an orderly fashion. The problem occurs because of the difference in the surfaces involved.

The model of the earth is either a sphere or spheroid (Chapter 3). These curved surfaces have two finite radii of curvature. The map is a plane surface, and a plane is characterized by two infinite radii of curvature. As will be shown in Chapter 2, it is impossible to transform from a surface of two finite radii of curvature to a surface of two infinite radii of curvature without introducing some distortion. The sphere and the spheroid are called

*Numbers in brackets refer to the bibliography.

nondevelopable surfaces. This refers to the inability of these surfaces to be developed (i.e., transformed) into a plane in a distortion free manner [8].

Intermediate between the nondevelopable sphere and the spheroid, and the plane are surfaces with one finite and one infinite radius of curvature. The examples of this type of figure are the cylinder and the cone. These surfaces are called developable. Both the cylinder and the cone can be cut, and then developed (essentially unrolled along the finite radius of curvature) to form a plane. This development introduces no distortion, and thus, these figures may be used as intermediate plotting surfaces between the sphere and spheroid, and the plane. However, in any transformation from the sphere or spheroid to the developable surface, the damage has already been done. The transformation from the nondevelopable to the developable surface has already introduced some degree of distortion.

Consider of what an ideal map would consist [8].

(1) Areas on the map would maintain correct proportion to areas on the earth.

(2) Distances on the map would remain in true scale.

(3) Directions and angles on the map would remain true.

(4) Shapes on the map would be the same as on the earth.

The impossibility of a distortion free transformation from the nondevelopable surface to the plane prevents the realization of the ideal. The best a cartographer can hope for is a realization of one or two of these features over the entire map. The other features are subject to distortion, but hopefully to a controlled extent.

The projections of Chapters 4, 5, and 6 are the cartographer's answers to the problems. In each of these projections, some of the desired features are maintained. The distortion in the other features will be tolerable.

1.2 Distortions [22]

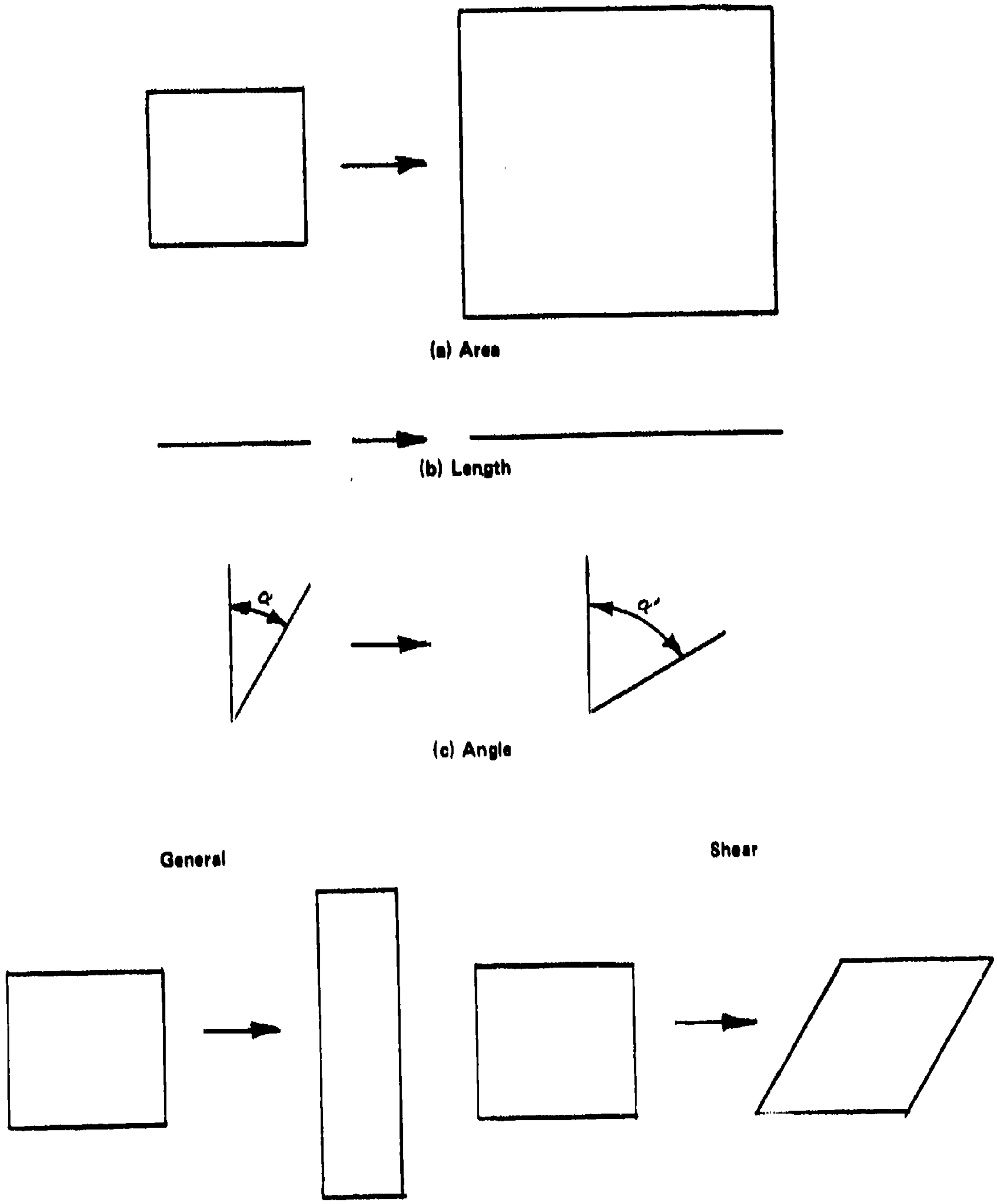
Distortion is the villain of the piece. Distortion in maps may be in area, length, angle, or shape.

Distortion in area is shown in Figure 1.2.1(a). While shape is maintained, the area on the map may be enlarged or diminished.

Distortion in length is common, and Figure 1.2.1(b) is an illustration. Often, while the cartographer is able to maintain true length in one direction, he cannot do so in a second direction.

Angular distortion is also prevalent. Thus, angles on a map will not necessarily be the same as their counterparts on the earth. Thus, azimuths on the map, α' will not coincide with true azimuths' α on the earth. This is shown in Figure 1.2.1(c).

Distortion in shape can occur in a number of ways. One is a general change of shape of the figure. A second is a shearing type of effect. Figure 1.2.1(d) demonstrates both of these changes.



(d) Shape

Figure 1.2.1. Distortion effects

An actual map will have combinations of these distortions. The numerical theory of distortion will be presented in Chapter 7 of this report.

Now that we are acquainted with the problem, the next sections in this chapter will introduce some of the terms needed for the study of map projections before entering the mathematics of Chapter 2.

1.3 Coordinate Systems [18]

Coordinate systems are necessary for both the earth and maps for the orderly location of points. Two types of coordinate systems will be considered for the earth. They are a Cartesian system, and an angular system. For maps, the most convenient system is Cartesian.

The terrestrial coordinate system is demonstrated in Figure 1.3.1. The origin, O, of the system is at the center of the earth. The x- and y-axis form the equatorial plane. The curve on the earth formed by the intersection of this plane with the earth's surface is the equator. The positive x-axis intersects the curve AGN. The curve AGN is a plane curve, which is called the Greenwich meridian. The positive z-axis coincides with the nominal axis of rotation of the earth, and points in the direction of the north pole, N. The y-axis completes a right-handed coordinate system.

Any point, P, on the surface of the earth can be located by the coordinates x, y, and z. However, since any point is constrained to lie on the surface, the three coordinates are not all independent. They are related by the equation of the surface (Chapter 3). Thus, there are only two independent coordinates, or two degrees of freedom.

Instead of using two arbitrarily chosen members of the set x, y, and z as the independent coordinates, it is more convenient to use two independent angular coordinates: latitude and longitude.

A meridian is a curve formed by the intersection of a fictitious plane containing the z-axis and the surface of the earth. The Greenwich meridian has already been mentioned. There is an infinity of meridians, depending on the orientation of the cutting plane.

The use of latitude and longitude depends on locating a point on a meridian and then locating the meridian with respect to the Greenwich meridian. Latitude is the angular measure defining the position of point P on the meridian BPN. Latitude is denoted by ϕ. The position of the meridian that contains P is defined by the longitude, λ. The longitude is the angle AOB, measured in the equatorial plane, from the Greenwich meridian.

The conventions for latitude and longitude are as follows. Latitude is measured plus to the north, and minus to the south. Longitude is measured positive to the east, and negative to the west.

The circles of parallel are generated by cutting planes parallel to the equatorial plane which intersect the earth. All points on the circle of parallel have the same latitude.

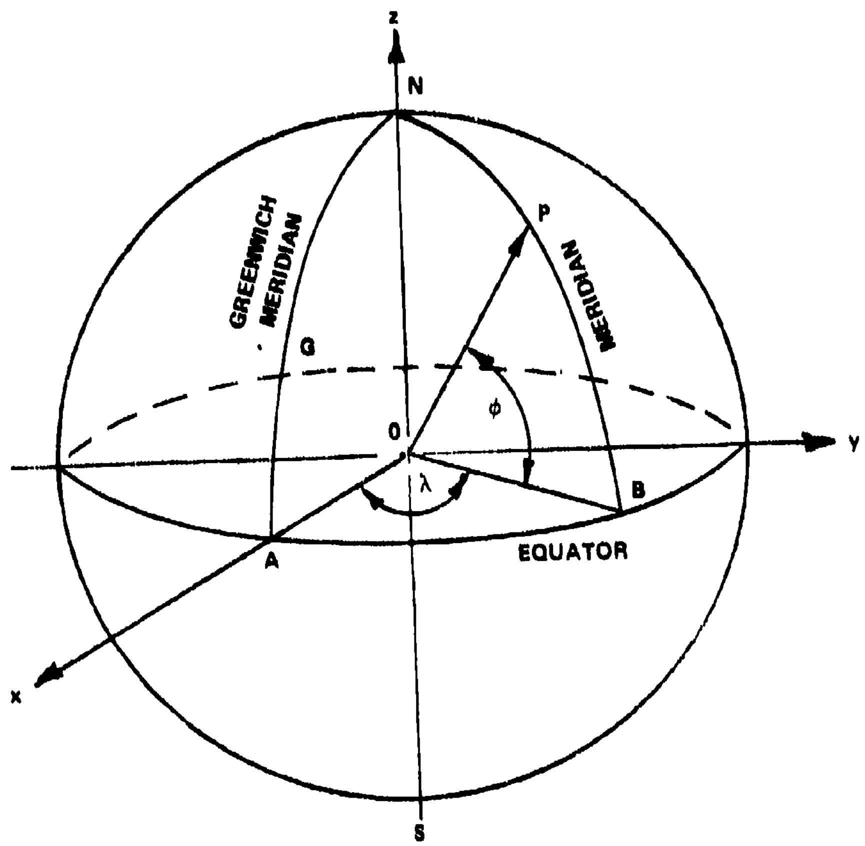


Figure 1.3.1. Terrestrial coordinate system

The mathematical relationships between the polar and Cartesian coordinates, as well as the definition of the types of latitude is deferred until Chapter 3, where the sphere and the spheroid are discussed.

The coordinate system for the map, Figure 1.3.2, is a two dimensional Cartesian system. The plus x-axis is toward the east, and the plus y-axis is toward the north. The origin, 0′, of the system will depend on the scheme of projection to be developed in Chapters 4, 5, and 6. In most cases there will be some straight, arbitrarily chosen central meridian which serves as the ordinate of the projection.

The object of map projection is to transform from the terrestrial angular system to the map Cartesian system. Chapters 4, 5, and 6 will provide the methods for these transformations.

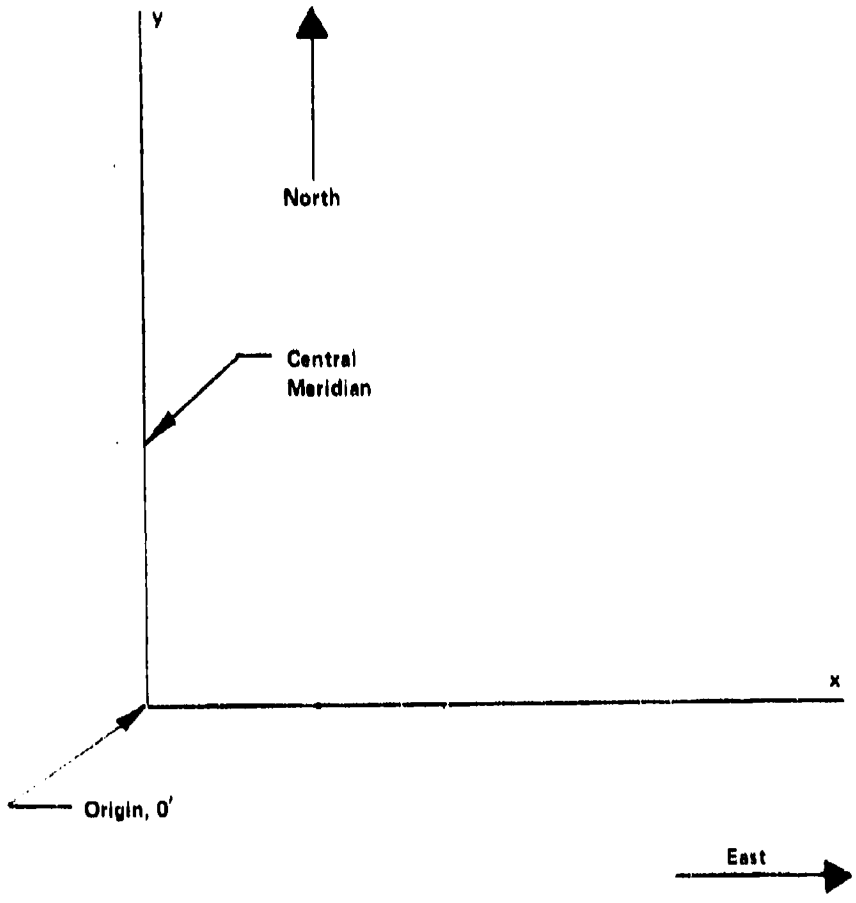


Figure 1.3.2. Map coordinate system

1.4 Scale [8]

The scale of the map is the ratio of the distance on the map to the corresponding distance on the earth, or dm/de. If the distance on both the map and the earth have the same units, then the scale is a dimensionless quantity. Scale is another aspect of the orderly transformation from earth measurement to map measurement.

The presence of distortion requires the definition of two types of scale: the principle scale and the local scale.

The principle scale is based on a meridian or parallel which is a uniformly true scale for the entire map. It is the scale used for shrinking the spheroidal surface of the earth to the plane of the paper.

At other places on the map, where distortions are present, the scale will be different from true scale. This local scale will be larger or smaller than the principal scale, depending on the mechanism of the distortion.

The local scale, as a function of distortion, and the principle scale may be quoted on the legend of a map. A more useful means is a graphical scale drawn in the map legend, and specified for the latitudes and longitudes where it applies. As an example, for the Mercator projection (Chapter 5), a set of scales can be drawn as a function of latitude, which will ensure the correct distances for measuring.

The terms large scale versus small scale come from consideration of the fraction dm/de. A scale of 1/10,000 is a large scale, and 1/1,000,000 is a small scale. A plan, or a map showing buildings, cultural features, or boundaries is usually 1/10,000 or larger. A topographic map, which gives roads, railroads, towns, and contour lines, and other details has a scale between 1/10,000 and 1/1 000,000. Maps of a scale smaller than 1/1,000,000 are atlas maps. These maps delineate countries, continents, and oceans.

The scale factor, S, is used in the plotting equations of Chapters 4, 5, and 6, and is equal to dm/de.

1.5 Classification by Feature Preserved [22]

Maps may be classified by the feature rescued from distortion, or by the agreement that some distortion will simply be tolerated. This system of classification divides maps into three catagories: equal area, conformal and conventional.

The equal area projection preserves the ratio of areas on the earth and on the map as a constant. Any part of the map bears the same relation to the area on the earth it represents that the whole map bears to the total earth area represented. Any quadrangular shaped section of the map formed by a grid of meridians and parallels will be equal in area to any other quadrangular area of the same map that represents an equal area of the earth. Angles usually suffer. A contraction of meridians will have to be offset by a lengthening of parallels, or vice versa, but the enclosed area will remain the same. This concept is illustrated in Figure 1.5.1(a). All of the quadrilaterals have the same area.

A conformal projection is one in which the shape of any small surface of the map is preserved in its original form. Care must be used in applying this concept, since it is true only locally, and cannot be extended over large surface areas. The true condition for a conformal map is that the scale at any point is the same in all directions. The scale will change from point to point, but it will be independent of the azimuth at all points. The scale will be the same in all directions from a point if two directions at right angles on the earth are mapped into two directions that are also at right angles to each other. The meridians and parallels of the earth intersect at right angles, and a conformal projection preserves this quality on the map. Conformal quadrilaterals are shown in Figure 1.5.1(b). Another term used in refering to conformal projections is orthomorphic, or same form.

Conventional projections are all those which are neither equal area nor conformal. This is not meant as a disparaging term. Many of the conventional maps are of great utility. In the Gnomonic projection, the feature preserved is that great circles become straight lines. In the Azimuthal equidistant projection the distance and azimuth from the origin to any other point on the map is true. The Polyconic and van der Grinten projections have seen considerable service as road maps. All that is implied by the term conventional is that the cartographer has been willing to sacrifice the features of equal area or conformality in order to retain some other feature, or to obtain a simple, utilitarian algorithm for the projection.

1.6 Classification by Projection Surface [22]

Only three projection surfaces will be considered – the plane, the cone, and the cylinder. All projections in use today are accomplished through these, or modifications of these. It can be argued that all projection surfaces are conical, since the plane and the cylinder can be considered as the two limiting cases of the cone. However, this mathematical nicety is not usually used, and the three surfaces will be considered as distinct, in most cases. Figure 1.6.1 shows each of these surfaces in relation to the sphere.

The planar projection surface can be used for a direct transformation from the earth. The projections which result are called azimuthal (Figure 1.6.1(a)). Other names in use are zenithal, or planar projections.

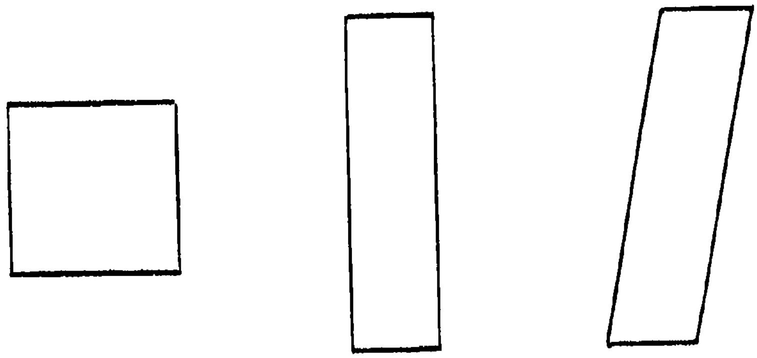
Conical projections result when a cone is used as an intermediate plotting surface. The cone is then developed into a plane to obtain the map (Figure 1.6.1(b)).

At this point it is convenient to introduce the concept of the constant of the cone. Let a be the radius of the earth. From Figure 1.6.2, the slant height of the cone tangent to the earth, ρ, is found to be

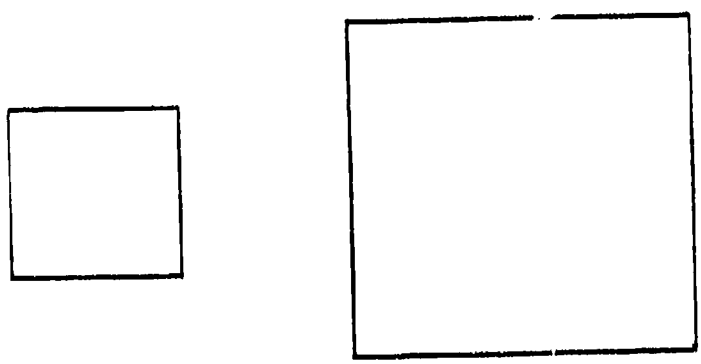
$$\rho = a \cot \phi \tag{1.6.1}$$

where ϕ is the latitude. Also, from the figure, d, the length of the parallel circle AB, which defines the circle of tangency of the cone, is

$$d = 2\pi a \cos \phi . \tag{1.6.2}$$

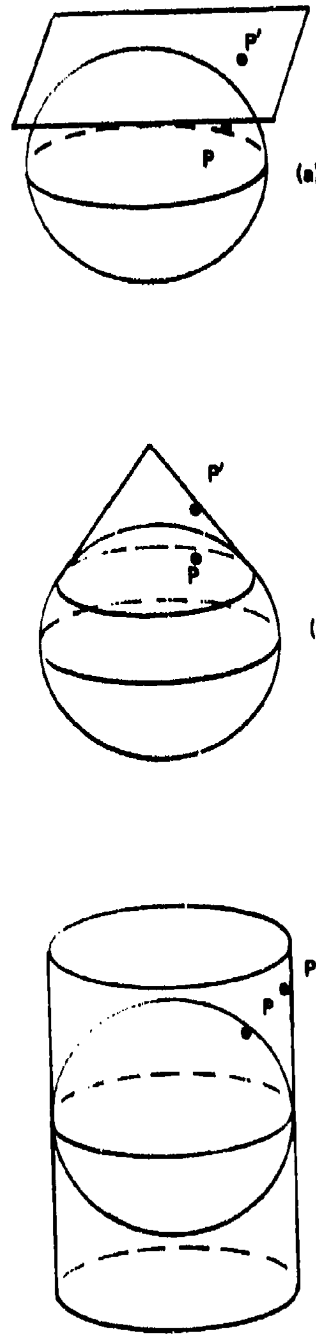


(a) Equal area quadrilaterals



(b) Conformal quadrilaterals

Figure 1.5.1. Quadrilateral representation



(a) Azimuthal

(b) Conical

(c) Cylindrical

Figure 1.6.1. Classification by projection surface

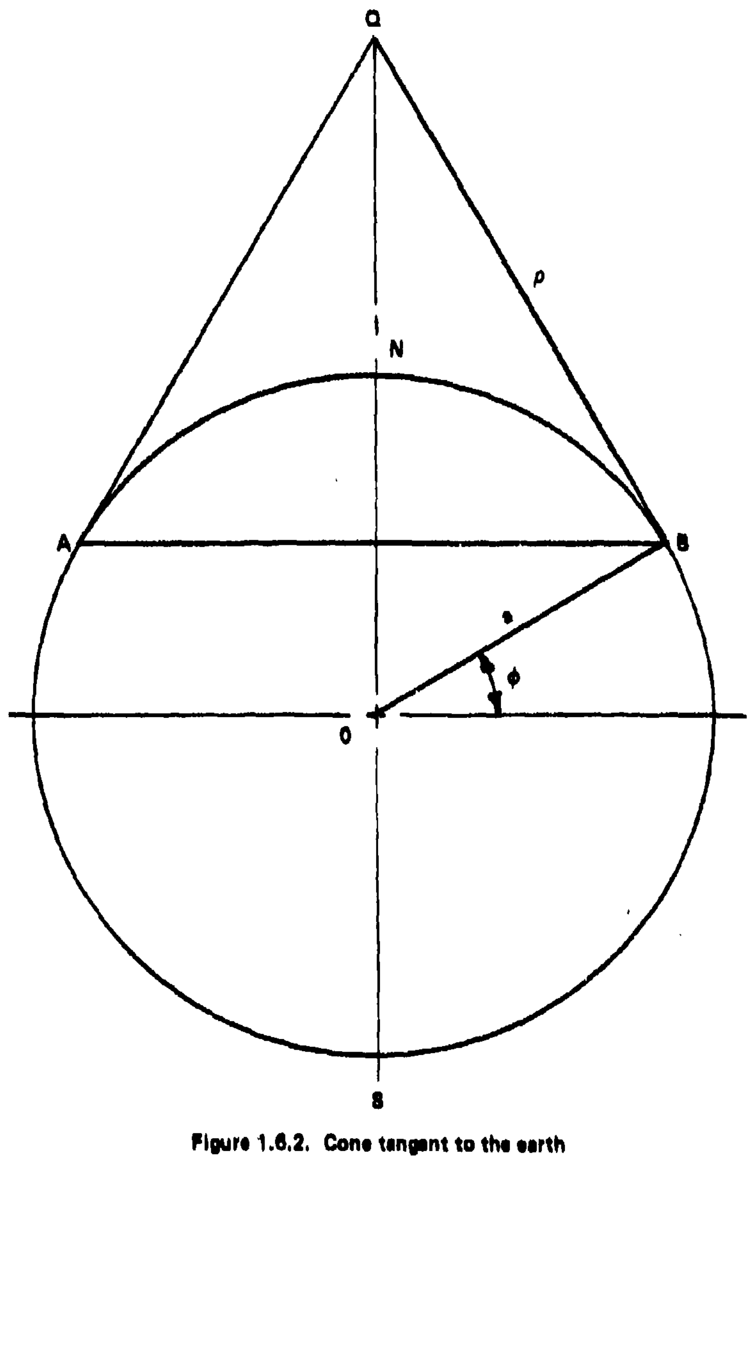


Figure 1.6.2. Cone tangent to the earth

The constant of the cone, c, is defined from the relation between the developed cone and the earth. Let,

$$\theta = d/\rho . \tag{1.6.3}$$

Substitute (1.6.1) and (1.6.2) into (1.6.3)

$$\theta = \frac{2\pi a \cos\phi}{a \cot\phi}$$

$$\theta = 2\pi \sin\phi . \tag{1.6.4}$$

The constant of the cone is $c = \sin\phi$. It is a multiplicative factor that relates longitudes on the earth to those on the cone. Equations (1.6.1) and (1.6.4) will be beneficial in Chapters 4, 5, and 6 in the investigation of the various conical projections.

The cone may also be secant to the earth. This is shown in Figure 1.6.3, where the circles of secancy are at the latitudes ϕ_1, and ϕ_2. From the similar triangles, the ratios of the slant heights of these respective latitudes are

$$\frac{\rho_1}{\rho_2} = \frac{a \cos\phi_1}{a \cos\phi_2}$$

$$= \frac{\cos\phi_1}{\cos\phi_2} . \tag{1.6.5}$$

Note in equation (1.6.4) that as ϕ varies from 0° to 90°, θ varies from 0° to 360°. When θ is 0°, then we have a cylinder. At θ equals 360°, we have a plane. As was mentioned above, it will be useful, usually, to treat planes, cones and cylinders as separate entities, rather than lump them together in a single general approach to the problem.

Cylindrical projections are obtained when a cylinder is used as the intermediate plotting surface (Figure 1.6.1(c)). As with the cone, the cylinder can then be developed into a plane.

In Figure 1.6.1, a representative position on the earth, P, is shown transformed into a position on the projection surface, P' for each of the projection surfaces. Chapters 4, 5, and 6 will explain the methods that will affect such transformations, and produce useful maps.

1.7. Classification by Orientation of the Azimuthal Plane [22]

As was seen in the previous section, the option of using a tangent cone or a secant cone is a means of further differentiating conical projections. Similarly, azimuthal projections may be classified by reference to the point of contact of the plotting surface with the earth.

Azimuthal projections may be classified as polar, equatorial, or oblique. When the plane is tangent to the earth at either pole, we have a polar projection. When the plane is tangent

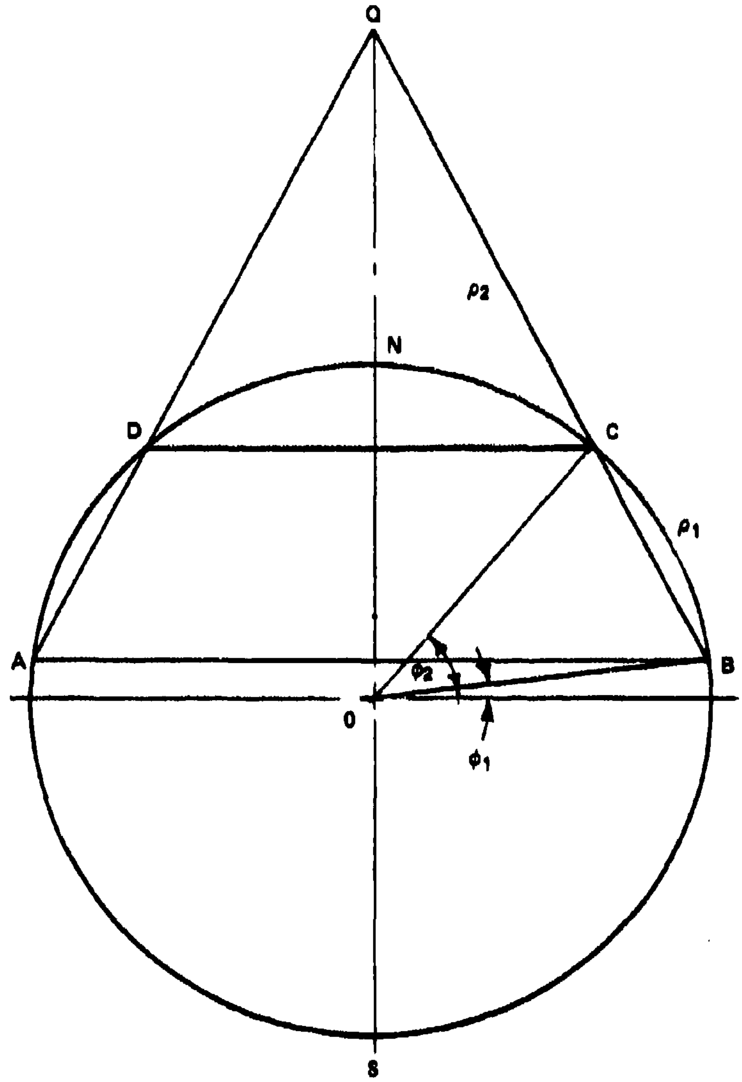


Figure 1.6.3. Cone secant to the earth

to the earth at any point on the equator, the projection is called equatorial. The oblique case occurs when the plane is tangent at any point on the earth except the poles and the equator. Figure 1.7.1 indicates these three alternatives. In each case, T is the point of tangency.

1.8 Classification by Orientation of a Cone or Cylinder [22]

Another classification set can be defined for cones and cylinders. These plotting surfaces may be considered to be regular, transverse, or oblique.

The regular projection occurs when the axis of the cone or cylinder coincides with the polar axis of the earth. The transverse case has the axis of the cone or cylinder perpendicular to, and intersecting the axis of the earth. The transverse Mercator, and the transverse Polyconic are examples of this. If the axis of the cone or cylinder has any other position in space, besides being coincident with, or perpendicular to, the axis of the earth, then an oblique projection is generated. Figure 1.8.1 demonstrates these three options for a simple projection.

1.9 Projection Technique [3], [11]

Three techniques of projection can be identified. This can serve as another scheme of classification. The methods are the graphical, the semi-graphical, and the mathematical.

In any graphical technique, some point 0 is chosen as a projection point, and the methods of projective geometry and descriptive geometry are used to transform a point P on the earth to a location P′ on the plotting surface. An example of this is indicated in Figure 1.9.1, where the point P on the earth is transformed to the oblique plane by the extension of line OP until it intersects the plane. In this example, 0 is arbitrarily chosen as the projection point. Since anything that can be done graphically can also be described mathematically, we will not encourage graphical constructions. However, those projections which are capable of a strict graphical approach will be identified in Chapters 4, 5, and 6.

Those projections termed mathematical will be those which can only be produced by a mathematical definition. No draftsman with compass and straight edge can plot them by means of the projection of a ray.

In between these two groups are the semi-graphical projections. However, for various reasons, such as a varying projection point (Mercator), or a complex graphical scheme (Mollweide), the reasonable approach is to depend on a mathematical procedure.

1.10 Azimuth [7]

The angular measure of use in specifying directions on the earth and on the map is the azimuth. The azimuth of P′ in relation to P is shown in Figure 1.10.1 for the earth. Azimuth is measured from the north, or the meridian through the point P, in a clockwise manner. Azimuth is measured the same way on the map as on the earth.

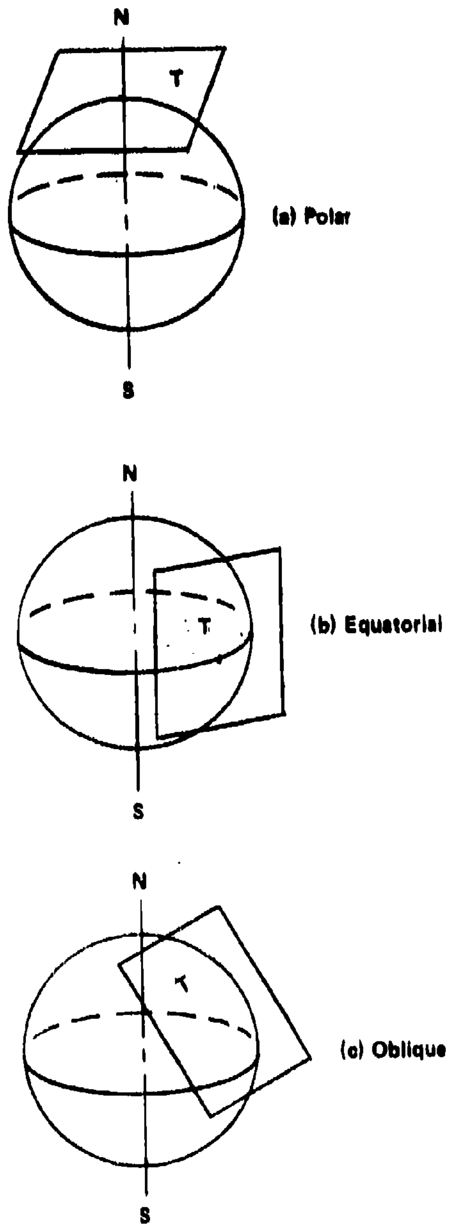


Figure 1.7.1. Orientation of the azimuthal plane

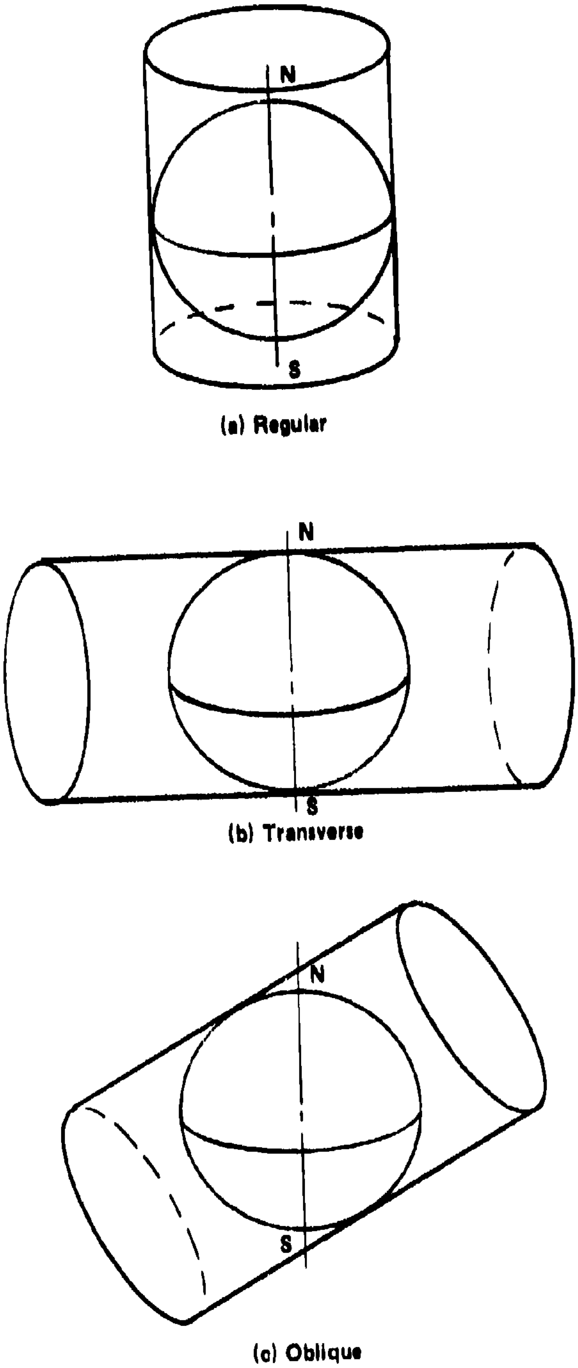


Figure 1.8.1. Orientation of a cylinder

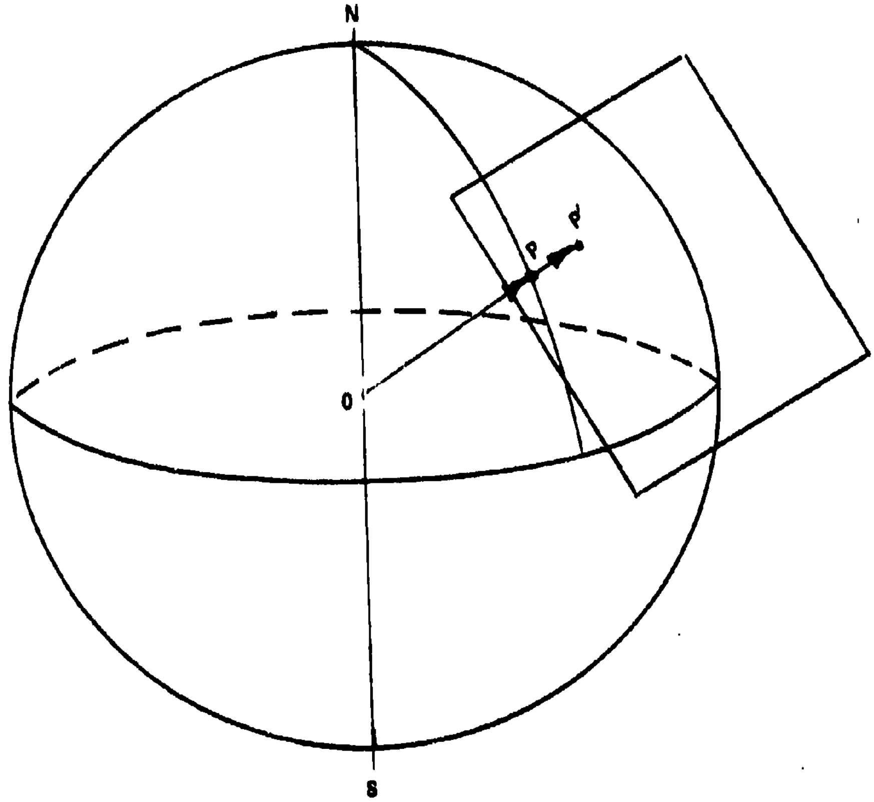


Figure 1.9.1. Graphical projection onto a plane

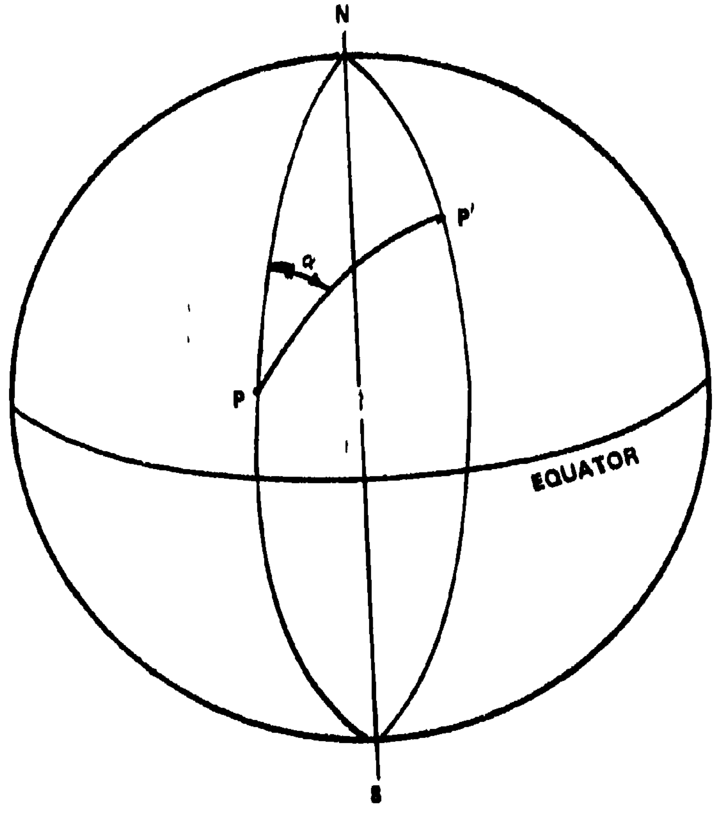


Figure 1.10.1. Azimuth of P' from P

1.11 Computer Implementation

The subject of map projections certainly has intrinsic interest. Of more importance to surveyors, cartographers, and all other workers in the field of map projections, is the availability of plotting equations, and their incorporation onto a computer program that permits their utilization. The ultimate goal of this report is to provide Cartesian plotting equations as a function of latitude and longitude, and a computer program which includes the most important of them. This will be, at the very least, a good beginning for further practical work.

After the basic concepts are derived in Chapters 2 and 3, the actual equations for mapping will be developed in Chapters 4, 5, and 6.

The computer program in Appendix A.1 combines twenty of the most useful map projections schemes with the methods of providing equal area and conformal qualities, and the methods of rotational transformation. In all cases, the latitude and longitude coordinates of the earth will be input, and converted into x-, and y-plotting coordinates.

These subroutines can be incorporated into existing and proposed programs. The output can be suitable for plotting tables, digital/analog plotters, or CRT display [12], [13].

Appendix A.1 includes an input guide to the computer program MAP. This will aid the user in incorporating his values of the earth parameters and scale factor, and selecting the required projection scheme. The method of selecting a complete grid or a collection of points is also described.

Chapter 2

MAPPING TRANSFORMATIONS

The process of map projection requires the transformation from the two independent coordinates of the earth to the two independent coordinates of the map. This chapter will be devoted to the general theory of transformations. To this end, it will be necessary to develop some applicable formulas of differential geometry, and apply some aspects of spherical trigonometry.

The differential geometry of curves will give the needed radius of curvature and torsion of a space curve. The differential geometry of surfaces will concern the first and second fundamental forms, and parametric curves and the condition of orthogonality. The surfaces of interest in mapping are surfaces of revolution. The general surface will be particularized to surfaces of revolution (Chapter 3). The process of transformation from non-developable to developable surfaces will be considered. Representations of arc length, angles, and area, as well as the definition of the normal to the surface, will be given.

The basic transformation matrix will be derived. The conditions of equal area and conformality will be applied to this transformation.

The convergence of the meridians is next considered. Finally, a rotation method for the production of equatorial, transverse, and oblique projections will be given.

2.1 Differential Geometry of Curves [10]

Consider the space curve of Figure 2.1.1. Let ζ be an arbitrary parameter. Let the vector to any point P, on the curve, in the Cartesian coordinate system, be

$$\mathbf{r} = x(\zeta)\hat{\imath} + y(\zeta)\hat{\jmath} + z(\zeta)\hat{k}. \tag{2.1.1}$$

Let $|\Delta \mathbf{r}| = \Delta s$.

The unit tangent vector at point P is

$$\underset{\Delta s \to 0}{\mathrm{L}} \frac{\Delta \mathbf{r}}{\Delta s} = \frac{d\mathbf{r}}{ds} = \hat{t}. \tag{2.1.2}$$

Applying the chain rule to (2.1.2), one finds

$$\hat{t} = \frac{d\mathbf{r}}{d\zeta}\frac{d\zeta}{ds}. \tag{2.1.3}$$

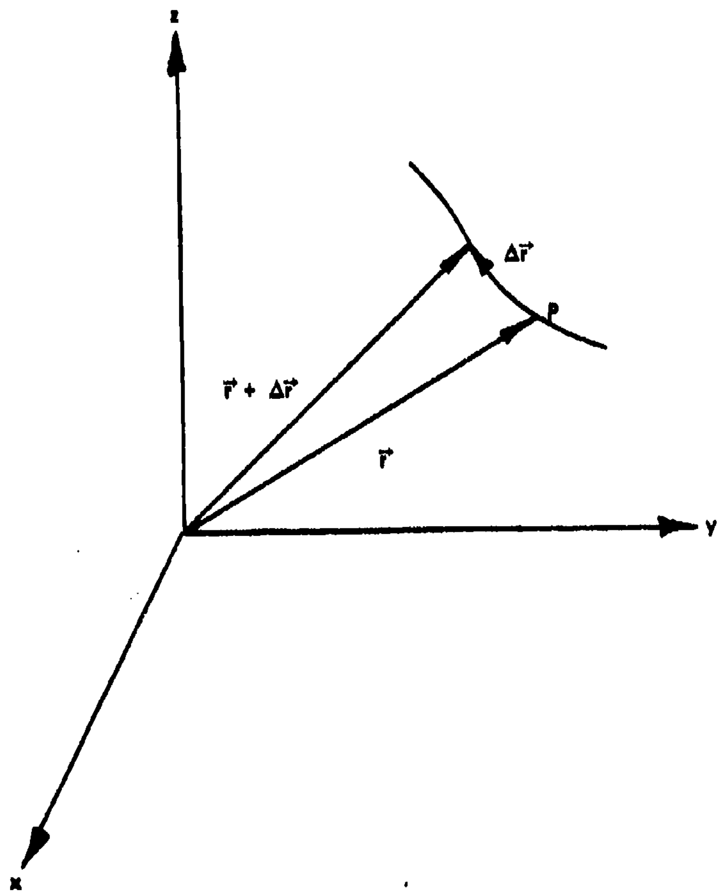


Figure 2.1.1. Geometry of a space curve

Taking the total differential of (2.1.3), we note

$$\hat{t} = \left(\frac{\partial x}{\partial \zeta}\,\hat{\imath} + \frac{\partial y}{\partial \zeta}\,\hat{\jmath} + \frac{\partial z}{\partial \zeta}\,\hat{k}\right)\left(\frac{d\zeta}{ds}\right). \tag{2.1.4}$$

Take the dot product of $\hat{t}$ with itself.

$$\hat{t} \cdot \hat{t} = 1 = \left[\left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2\right]\left(\frac{d\zeta}{ds}\right)^2$$

$$\left(\frac{d\zeta}{ds}\right)^2 = \frac{1}{\left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2}$$

$$\frac{d\zeta}{ds} = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 + \left(\frac{\partial z}{\partial \zeta}\right)^2}}$$

$$= \frac{1}{\left|\frac{\partial r}{\partial \zeta}\right|}. \tag{2.1.5}$$

Next, look at two consecutive tangent vectors, as shown in Figure 2.1.2.

$$\underset{\Delta s \to 0}{L} \frac{\Delta \hat{t}}{\Delta s} = \frac{d\hat{t}}{ds}$$

let

$$\frac{d\hat{t}}{ds} = -k\hat{n} \tag{2.1.6}$$

where k is defined as the curvature, and $\hat{n}$ is the principal unit normal.

Dot $\hat{t}$ with itself, and differentiate.

$$\hat{t} \cdot \hat{t} = 1$$

$$2\hat{t} \cdot \frac{d\hat{t}}{ds} = 0\,.$$

This means that $\hat{t}$ is perpendicular to $d\hat{t}/ds$, and, from (2.1.6), $\hat{n}$ is perpendicular to $\hat{t}$.

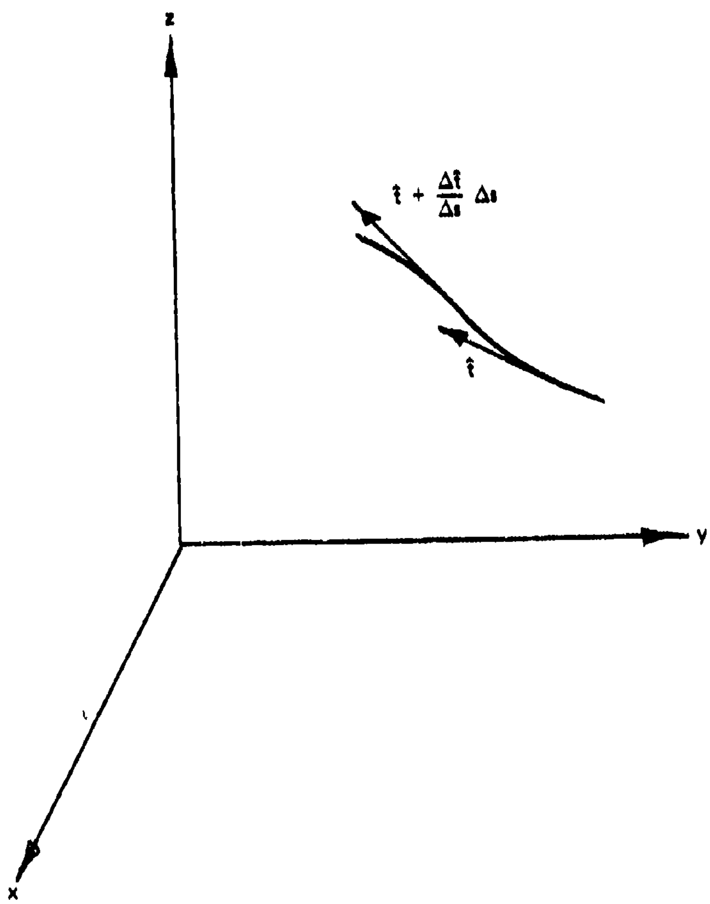


Figure 2.1.2. Consecutive tangent vectors

In order to obtain a right-handed triad, define the binormal vector

$$\hat{b} = \hat{t} \times \hat{n} \, . \tag{2.1.7}$$

Thus, we have $\hat{t}$, $\hat{n}$, and $\hat{b}$ as the unit vectors at P.

It is useful at this time to define three types of planes intersecting the curve. These are the osculating, normal, and rectifying planes. The osculating plane contains $\hat{t}$ and $\hat{n}$. The normal plane contains $\hat{n}$ and $\hat{b}$. The rectifying plane is defined by $\hat{t}$ and $\hat{b}$. These planes are displayed in Figure 2.1.3.

It is useful now to obtain the derivatives of the unit vectors as a function of distance along the curve. From the definition of the unit vectors, we have the relations

$$\left.\begin{aligned} \hat{b} &= \hat{t} \times \hat{n} \\ \hat{t} &= \hat{n} \times \hat{b} \\ \hat{n} &= \hat{b} \times \hat{t} \end{aligned}\right\} . \tag{2.1.8}$$

From the first of (2.1.8)

$$\begin{aligned} \frac{d\hat{b}}{ds} &= \frac{d}{ds}\,(\hat{t} \times \hat{n}) \\ &= \frac{d\hat{t}}{ds} \times \hat{n} + \hat{t} \times \frac{d\hat{n}}{ds} \, . \end{aligned} \tag{2.1.9}$$

Substitute (2.1.6) into (2.1.9).

$$\begin{aligned} \frac{d\hat{b}}{ds} &= -k\hat{n} \times \hat{n} + \hat{t} \times \frac{d\hat{n}}{ds} \\ &= \hat{t} \times \frac{d\hat{n}}{ds} \, . \end{aligned} \tag{2.1.10}$$

Dot $\hat{n}$ with itself, and differentiate.

$$\hat{n} \cdot \hat{n} = 1$$

$$2\hat{n} \cdot \frac{d\hat{n}}{ds} = 0 \, .$$

Thus, $d\hat{n}/ds$ is perpendicular to $\hat{n}$, and must lie in the rectifying plane, and have the components

$$\frac{d\hat{n}}{ds} = \psi\hat{t} + \tau\hat{b} \, . \tag{2.1.11}$$

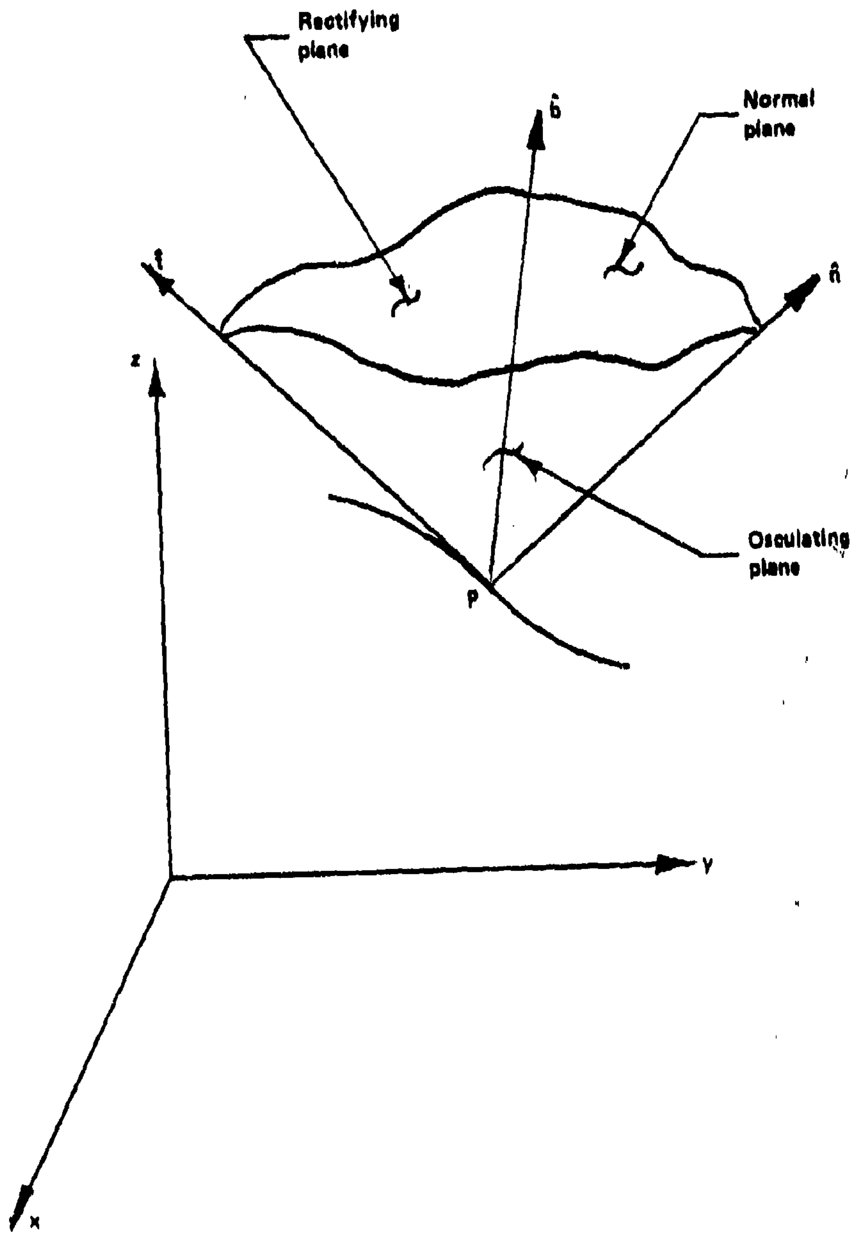


Figure 2.1.3. Planes on the space curve at point P

Substitute (2.1.11) into (2.1.10)

$$\frac{d\hat{b}}{ds} = \hat{t} \times (\psi t + \tau\hat{b})$$

$$= \tau\hat{t} \times \hat{b}$$

$$= -\tau\hat{n}\,. \tag{2.1.12}$$

The constant τ is called the torsion. It is, essentially, a measure of the twist of the curve.

From the last of (2.1.8)

$$\frac{d\hat{n}}{ds} = \frac{d}{ds}(\hat{b} \times \hat{t})$$

$$= \frac{d\hat{b}}{ds} \times \hat{t} + \hat{b} \times \frac{d\hat{t}}{ds}. \tag{2.1.13}$$

Substitute (2.1.6) and (2.1.12) into (2.1.13).

$$\frac{d\hat{n}}{ds} = -\tau\hat{n} \times \hat{t} + \hat{b} \times (-k\hat{n})$$

$$= \tau\hat{b} + k\hat{t}\,. \tag{2.1.14}$$

Equations (2.1.6), (2.1.13), and (2.1.14) can be arranged in matrix form.

$$\begin{bmatrix} d\hat{t}/ds \\ d\hat{n}/ds \\ d\hat{b}/ds \end{bmatrix} = \begin{bmatrix} 0 & -k & 0 \\ k & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{bmatrix}. \tag{2.1.15}$$

These are the Frenet-Serret formulas.

The next step will be to obtain the mathematical relations for the curvature and the torsion.

The curvature, in general parametric form, is obtained from (2.1.2) and (2.1.6).

$$
\begin{aligned}
-k\hat{n} &= \frac{d\hat{t}}{ds} \\
&= \frac{d}{ds}\left(\frac{dr}{ds}\right) \\
&= \frac{d^2 r}{ds^2}
\end{aligned}
\tag{2.1.16}
$$

Take the cross product of $\hat{t}$ with (2.1.16)

$$
\hat{t} \times (-k\hat{n}) = \hat{t} \times \left(\frac{d^2 r}{dt^2}\right). \tag{2.1.17}
$$

Apply the first of (2.1.8), and (2.1.2) to (2.1.17).

$$
-k\hat{b} = \frac{dr}{ds} \times \frac{d^2 r}{ds^2}
$$

$$
k = \left| \frac{dr}{ds} \times \frac{d^2 r}{ds^2} \right| . \tag{2.1.18}
$$

For the general parameterization,

$$
\frac{dr}{ds} = \frac{dr}{d\zeta}\,\frac{d\zeta}{ds} \tag{2.1.19}
$$

$$
\frac{d^2 r}{ds} = \frac{d^2 r}{d\zeta^2}\left(\frac{d\zeta}{ds}\right)^2 + \frac{dr}{d\zeta}\left(\frac{d^2\zeta}{ds^2}\right). \tag{2.1.20}
$$

Substitute (2.1.19) and (2.1.20) into (2.1.18).

$$
\begin{aligned}
k &= \left| \frac{dr}{d\zeta}\,\frac{d\zeta}{ds} \times \left[\frac{d^2 r}{d\zeta^2}\left(\frac{d\zeta}{ds}\right)^2 + \frac{dr}{d\zeta}\left(\frac{d^2\zeta}{ds^2}\right)\right] \right| \\
&= \left| \frac{dr}{d\zeta} \times \frac{d^2 r}{d\zeta^2} \right| \left(\frac{d\zeta}{ds}\right)^3 .
\end{aligned}
\tag{2.1.21}
$$

Substitute (2.1.5) into (2.1.21).

$$k = \frac{\left|\frac{dr}{d\xi} \times \frac{d^2 r}{d\xi^2}\right|}{\left|\frac{dr}{d\xi}\right|^3}. \tag{2.1.22}$$

A similar procedure can be followed to obtain the torsion. Dot (2.1.12) with $\hat{n}$.

$$\tau = -\frac{d\hat{b}}{ds} \cdot \hat{n}\,. \tag{2.1.23}$$

Substitute the first of (2.1.8) into (2.1.23).

$$\tau = -\frac{d}{ds}(\hat{t} \times \hat{n}) \cdot \hat{n}$$

$$= -\left(\frac{d\hat{t}}{ds} \times \hat{n} + \hat{t} \times \frac{d\hat{n}}{ds}\right) \cdot \hat{n}$$

$$= (\hat{t} \times \hat{n}) \cdot \frac{d\hat{n}}{ds}. \tag{2.1.24}$$

From (2.1.16)

$$\hat{n} = -\frac{\frac{d^2 r}{ds^2}}{k}$$

$$\frac{d\hat{n}}{ds} = -\frac{\frac{d^3 r}{ds^3}}{k} + \frac{\frac{dk}{ds}}{k^2}\,\frac{d^2 r}{ds^2}. \tag{2.1.26}$$

Substitute (2.1.2), (2.1.25), and (2.1.26) into (2.1.24).

$$\tau = -\left(\frac{dr}{ds} \times \frac{\frac{d^2 r}{ds^2}}{k}\right) \cdot \left(\frac{\frac{d^3 r}{ds^3}}{k} - \frac{\frac{dk}{ds}}{k^2}\,\frac{d^2 r}{ds^2}\right)$$

$$= \frac{1}{k^2}\left(\frac{dr}{ds} \times \frac{d^2 r}{ds^2}\right) \cdot \frac{d^3 r}{ds^3}. \tag{2.1.27}$$

For the general parameterization, differentiate (2.1.20)

$$\frac{d^3\mathbf{r}}{ds^3} = \frac{d^3\mathbf{r}}{d\zeta^3}\left(\frac{d\zeta}{ds}\right)^3 + 3\,\frac{d^2\mathbf{r}}{d\zeta^2}\left(\frac{d\zeta}{ds}\right)\left(\frac{d^2\zeta}{ds^2}\right)$$

$$+\frac{d\mathbf{r}}{d\zeta}\,\frac{d^3\zeta}{ds^3}. \tag{2.1.28}$$

Substitute (2.1.19), (2.1.20), and (2.1.28) into (2.1.2.1.27)

$$\tau = \frac{\left[\left(\frac{d\mathbf{r}}{d\zeta} \times \frac{d^2\mathbf{r}}{d\zeta^2}\right) \cdot \left(\frac{d^3\mathbf{r}}{d\zeta^3}\right)\right]\left(\frac{d\zeta}{ds}\right)^6}{k^2}. \tag{2.1.29}$$

Substitute (2.1.5) into (2.1.29).

$$\tau = \frac{\left[\left(\frac{d\mathbf{r}}{d\zeta} \times \frac{d^2\mathbf{r}}{d\zeta}\right) \cdot \left(\frac{d^3\mathbf{r}}{d\zeta^3}\right)\right]}{k^2\left|\frac{d\mathbf{r}}{d\zeta}\right|^6}. \tag{2.1.30}$$

The torsion is important for such curves as the geodesic (Chapter 3). For plane curves, such as the meridian curve, and the equator, $\tau = 0$.

As an example, consider a plane curve [16], and let $\zeta = x$, and $y = y(x)$. The radius vector is

$$\mathbf{r} = x\hat{\imath} + y(x)\hat{\jmath}. \tag{2.1.31}$$

Obtain the curvature, by differentiating (2.1.31).

$$\frac{d\mathbf{r}}{dx} = \hat{\imath} + \frac{dy}{dx}\,\hat{\jmath} \tag{2.1.32}$$

$$\frac{d^2\mathbf{r}}{dx^2} = \frac{d^2y}{dx^2}\,\hat{\jmath}. \tag{2.1.33}$$

Substitute (2.1.32) and (2.1.33) into (2.1.22).

$$k = \frac{\left|\left(\hat{\imath} + \frac{dy}{dx}\,\hat{\jmath}\right) \times \left(\frac{d^2y}{dx^2}\,\hat{\jmath}\right)\right|}{\left[\left(\hat{\imath} + \frac{dy}{dx}\,\hat{\jmath}\right) \cdot \left(\hat{\imath} + \frac{dy}{dx}\,\hat{\jmath}\right)\right]^{3/2}} \tag{2.1.34}$$

Continued

$$= \frac{\left|\frac{d^2y}{dx^2}\,\hat{k}\right|}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}} . \tag{2.1.34}$$

The radius of curvature is the reciprocal of the curvature. Thus, from (2.1.34, and taking the magnitude,

$$\rho = 1/k$$

$$= \left|\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}\right| . \tag{2.1.35}$$

Continuing from (2.1.33), we note that

$$\frac{d^3\mathbf{r}}{dx^3} = \frac{d^3y}{dx^3}\,\hat{j} . \tag{2.1.36}$$

Substitute (2.1.32), (2.1.33), and (2.1.36) into (2.1.30).

$$\tau = \frac{\begin{vmatrix} 1 & \frac{dy}{dx} & 0 \\ 0 & \frac{d^2y}{dx^2} & 0 \\ 0 & \frac{d^3y}{dx^3} & 0 \end{vmatrix}}{k^2\left|\hat{i} + \frac{dy}{dx}\,\hat{j}\right|^6} = 0 .$$

2.2 Differential Geometry of Surfaces [10]

The parametric representation of a surface requires two parameters. In general, for the parametric representation of a surface by two arbitrary parameters, α_1 and α_2, the vector to a point on the surface is

$$\mathbf{r} = \mathbf{r}(\alpha_1, \alpha_2) . \tag{2.2.1}$$

If either of the two parameters is held constant, and the other one is varied, a space curve results. This space curve is the parametric curve. Figure 2.2.1 gives the parametric representation of space curves on a surface. The α_1-curve is the parametric curve along which α_2 is constant, and the α_2-curve is the parametric curve along which α_1 is constant.

The next step is to obtain the tangents to the parametric curves at point P. The tangent vector to the α_1-curve is

$$\mathbf{a}_1 = \frac{\partial r}{\partial \alpha_1} . \tag{2.2.2}$$

The tangent to the α_2-curve is

$$\mathbf{a}_2 = \frac{\partial r}{\partial \alpha_2} . \tag{2.2.3}$$

The plane spanned by the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ is the tangent plane to the surface at point P.

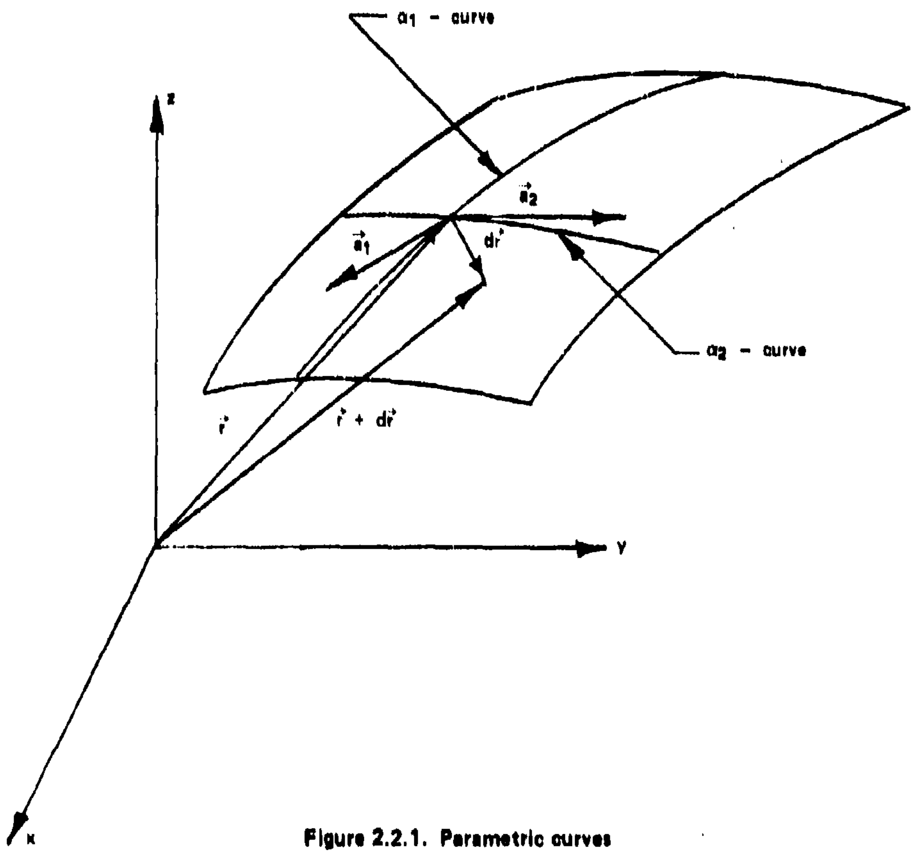


Figure 2.2.1. Parametric curves

The total differential of (2.2.1) is

$$dr = \frac{\partial r}{\partial \alpha_1}\, d\alpha_1 + \frac{\partial r}{\partial \alpha_2}\, d\alpha_2 \,. \tag{2.2.4}$$

Substituting (2.2.2) and (2.2.3) into (2.2.4),

$$dr = \mathbf{a}_1\, d\alpha_1 + \mathbf{a}_2\, d\alpha_2 \,. \tag{2.2.5}$$

Armed with equation (2.2.5), we are now ready to introduce the first fundamental form.

2.3 First Fundamental Form [10]

The first fundamental form of a surface is now to be derived. The first fundamental form is useful in dealing with arc length, area, angular measure on a surface, and the normal to the surface.

From (2.2.5)

$$\begin{aligned}(ds)^2 &= dr \cdot dr \\ &= (\mathbf{a}_1\, d\alpha_1 + \mathbf{a}_2\, d\alpha_2)(\mathbf{a}_1\, d\alpha_1 + \mathbf{a}_2\, d\alpha_2) \\ &= \mathbf{a}_1 \cdot \mathbf{a}_1 (d\alpha_1)^2 + 2(\mathbf{a}_1 \cdot \mathbf{a}_2)\, d\alpha_1\, d\alpha_2 \\ &\quad + \mathbf{a}_2 \cdot \mathbf{a}_2 (d\alpha_2)^2 \,.\end{aligned} \tag{2.3.1}$$

Define new variables,

$$\left.\begin{aligned} E &= \mathbf{a}_1 \cdot \mathbf{a}_1 \\ F &= \mathbf{a}_1 \cdot \mathbf{a}_2 \\ G &= \mathbf{a}_2 \cdot \mathbf{a}_2 \end{aligned}\right\} . \tag{2.3.2}$$

Substitute (2.3.2) into (2.3.1),

$$(ds)^2 = E(d\alpha_1)^2 + 2F d\alpha_1 d\alpha_2 + G(d\alpha_2)^2 \,. \tag{2.3.3}$$

Equation (2.3.3) is the first fundamental form of a surface, and this will be very useful through the whole process of map projection. The first fundamental form will now be applied to linear measure on any surface.

Arc length can be found immediately from the integration of (2.3.3). The distance between two arbitrary points P_1 and P_2 on the surface is given by

$$s = \int_{P_1}^{P_2} \sqrt{E(d\alpha_1)^2 + 2F d\alpha_1 d\alpha_2 + G(d\alpha_2)^2}$$

$$= \int_{P_1}^{P_2} \left\{ \sqrt{E + 2F\left(\frac{d\alpha_2}{d\alpha_1}\right) + G\left(\frac{d\alpha_2}{d\alpha_1}\right)^2} \right\} d\alpha_1 . \qquad (2.3.4)$$

Equation (2.3.4) is useful as soon as $d\alpha_2/d\alpha_1$ is defined, and will be used in Chapter 3 for distance along the spheroid.

Angles between two unit tangents $\mathbf{a}_1$ and $\mathbf{a}_2$ on the surface can be found by taking the dot product of (2.2.2) and (2.2.3) and applying (2.3.2).

$$\cos\theta = \frac{\mathbf{a}_1}{|\mathbf{a}_1|} \cdot \frac{\mathbf{a}_2}{|\mathbf{a}_2|} \qquad (2.3.5)$$

$$= \frac{F}{\sqrt{EG}} \qquad (2.3.6)$$

$$\sin\theta = \sqrt{1 - \cos^2\theta}$$

$$= \sqrt{1 - \frac{F^2}{EG}}$$

$$= \sqrt{\frac{EG - F^2}{EG}} . \qquad (2.3.7)$$

Define

$$H = EG - F^2 \qquad (2.3.8)$$

and substitute (2.3.8) into (2.3.7).

$$\sin\theta = \sqrt{\frac{H}{EG}} . \qquad (2.3.9)$$

The normal to the surface at point P is

$$\hat{n} = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|}$$

$$= \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1||\mathbf{a}_2| \sin\theta} . \qquad (2.3.10)$$

Substitute (2.3.2) and (2.3.8) into (2.3.9).

$$\hat{n} = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\sqrt{E}\sqrt{G}\sqrt{\frac{H}{EG}}}$$

$$= \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\sqrt{H}} . \qquad (2.3.11)$$

Incremental area can be obtained by a consideration of incremental distance along the parametric curves. Along the α_1-curve, and the α_1-curve, respectively,

$$\left.\begin{aligned} ds_1 &= \sqrt{E}\, d\alpha_1 \\ ds_2 &= \sqrt{G}\, d\alpha_2 \end{aligned}\right\} , \qquad (2.3.12)$$

The area is

$$dA = ds_1\, ds_2 \sin\theta$$

$$= \sqrt{EG}\, d\alpha_1\, d\alpha_2 \sin\theta . \qquad (2.3.13)$$

Substitute (2.3.8) into (2.3.13)

$$dA = \sqrt{EG}\sqrt{\frac{H}{EG}}\, d\alpha_1\, d\alpha_2$$

$$= \sqrt{H}\, d\alpha_1\, d\alpha_2 . \qquad (2.3.14)$$

Thus, the first fundamental form has given a means to derive the arc length, the unit normal to the surface at every point, and incremental area. In conjunction with the second fundamental form of the next section, it will be useful in determining the radii of curvature of the surface.

As will be shown in Chapter 3, the first fundamental form for the sphere is

$$(ds)^2 = a^2(d\phi)^2 + a^2\cos^2\phi(d\lambda)^2 \qquad (2.3.15)$$

and for the spheroid, it is

$$(ds)^2 = R_m^2(d\phi)^2 + R_p^2\cos^2\phi(d\lambda)^2 . \qquad (2.3.16)$$

When the chosen parameters are such as to ensure that the parametric curves are orthogonal to each other, a simplification of the first fundamental form occurs. When orthogonality is present, from (2.3.2), $\mathbf{a}_1$ and $\mathbf{a}_2$ are perpendicular, and $F = 0$. The first fundamental form is then

$$(ds)^2 = E(d\alpha_1)^2 + G(d\alpha_2)^2 . \qquad (2.3.17)$$

2.4 The Second Fundamental Form [10]

The second fundamental form provides a way to evaluate principal directions and curvatures of the surface. We will deal with normal sections through the surface, and derive formulas for the curvature of a normal section. A normal section implies that the normal to the parametric curve and the surface coincide.

We begin with a formula for curvature. For the parametric curve, take the dot product of $\hat{n}$ with (2.1.5).

$$\hat{n} \cdot \frac{d\hat{t}}{ds} = -k\hat{n} \cdot \hat{n}$$

$$= -k . \qquad (2.4.1)$$

Substitute the derivative of (2.1.2) into (2.4.1).

$$k = -\frac{d^2 r}{ds^2} \cdot \hat{n} . \qquad (2.4.2)$$

Since $\hat{t}$ and $\hat{n}$ are orthogonal,

$$\hat{t} \cdot \hat{n} = 0 . \qquad (2.4.3)$$

Substitute (2.1.2) into (2.4.3).

$$\frac{dr}{ds} \cdot \hat{n} = 0 . \qquad (2.4.4)$$

Take the derivative of (2.4.4).

$$\frac{d}{ds}\left(\frac{dr}{ds} \cdot \hat{n}\right) = 0$$

$$= \frac{d^2 r}{ds^2} \cdot \hat{n} + \frac{dr}{ds} \cdot \frac{d\hat{n}}{ds}$$

$$\frac{dr}{ds} \cdot \frac{d\hat{n}}{ds} = -\frac{d^2 r}{ds^2} \cdot \hat{n} . \qquad (2.4.5)$$

Substitute (2.4.2) into (2.4.5).

$$\frac{dr}{ds} \cdot \frac{d\hat{n}}{ds} = k\,. \tag{2.4.6}$$

From the total differential,

$$dr = \frac{\partial r}{\partial \alpha_1}\, d\alpha_1 + \frac{\partial r}{\partial \alpha_2}\, d\alpha_2 \tag{2.4.7}$$

$$d\hat{n} = \frac{\partial \hat{n}}{\partial \alpha_1}\, d\alpha_1 + \frac{\partial \hat{n}}{\partial \alpha_2}\, d\alpha_2\,. \tag{2.4.8}$$

Substitute (2.4.7) and (2.4.8) into (2.4.6).

$$k = \frac{\left(\frac{\partial r}{\partial \alpha_1}\, d\alpha_1 + \frac{\partial r}{\partial \alpha_2}\, d\alpha_2\right) \cdot \left(\frac{\partial \hat{n}}{\partial \alpha_1}\, d\alpha_1 + \frac{\partial \hat{n}}{\partial \alpha_2}\, d\alpha_2\right)}{(ds)^2}\,. \tag{2.4.9}$$

Substitute (2.2.2), (2.2.3), and (2.3.3) into (2.4.9).

$$k = \frac{\mathbf{a}_1 \cdot \frac{\partial \hat{n}}{\partial \alpha_1}\,(d\alpha_1)^2 + \mathbf{a}_2 \cdot \frac{\partial \hat{n}}{\partial \alpha_1}\,(d\alpha_2)^2 + \left(\mathbf{a}_1 \cdot \frac{\partial \hat{n}}{\partial \alpha_2} + \mathbf{a}_2 \cdot \frac{\partial \hat{n}}{\partial \alpha_1}\right) d\alpha_1\, d\alpha_2}{E(d\alpha_1)^2 + 2F d\alpha_1\, d\alpha_2 + G(d\alpha_2)^2}\,. \tag{2.4.10}$$

The second fundamental form is defined as

$$\mathbf{a}_1 \cdot \frac{\partial \hat{n}}{\partial \alpha_1}\,(d\alpha_1)^2 + \mathbf{a}_2 \cdot \frac{\partial \hat{n}}{\partial \alpha_2}\,(d\alpha_2)^2 + \left(\mathbf{a}_1 \cdot \frac{\partial \hat{n}}{\partial \alpha_2} + \mathbf{a}_2 \cdot \frac{\partial \hat{n}}{\partial \alpha_1}\right) d\alpha_1\, d\alpha_2\,. \tag{2.4.11}$$

Thus (2.4.10) is

$$k = \frac{\text{Second fundamental form}}{\text{First fundamental form}}\,.$$

It remains to define the coefficients of the differentials in the second fundamental form. From the definitions of the tangent and normal vectors

$$\hat{n} \cdot \mathbf{a}_i = \hat{n} \cdot \frac{\partial r}{\partial \alpha_i} = 0\,. \tag{2.4.12}$$

Take the derivative of (2.4.12)

$$\frac{\partial}{\partial \alpha_j}\left(\hat{n} \cdot \frac{\partial \mathbf{r}}{\partial \alpha_i}\right) = 0$$

$$\frac{\partial \hat{n}}{\partial \alpha_j} \cdot \frac{\partial \mathbf{r}}{\partial \alpha_i} + \hat{n} \cdot \frac{\partial^2 \mathbf{r}}{\partial \alpha_i \partial \alpha_j} = 0\,. \tag{2.4.13}$$

By definition, the second fundamental quantities are

$$b_{ij} = \frac{\partial \hat{n}}{\partial \alpha_j} \cdot \frac{\partial \mathbf{r}}{\partial \alpha_i}. \tag{2.4.14}$$

Substitute (2.4.14) into (2.4.13).

$$b_{ij} = -\hat{n} \cdot \frac{\partial^2 \mathbf{r}}{\partial \alpha_i \partial \alpha_j}. \tag{2.4.15}$$

Substitute (2.3.9) into (2.4.15).

$$\begin{aligned} b_{ij} &= -\frac{\mathbf{a}_1 \times \mathbf{a}_2}{\sqrt{H}} \cdot \frac{\partial^2 \mathbf{r}}{\partial \alpha_i \partial \alpha_j} \\ &= -\frac{1}{\sqrt{H}}\left(\frac{\partial^2 \mathbf{r}}{\partial \alpha_i \partial \alpha_j} \times \mathbf{a}_1\right) \cdot \mathbf{a}_2 \\ &= -\frac{1}{\sqrt{H}} \begin{vmatrix} \dfrac{\partial^2 x}{\partial \alpha_i \partial \alpha_j} & \dfrac{\partial^2 y}{\partial \alpha_i \partial \alpha_j} & \dfrac{\partial^2 z}{\partial \alpha_i \partial \alpha_j} \\ \dfrac{\partial x}{\partial \alpha_1} & \dfrac{\partial y}{\partial \alpha_1} & \dfrac{\partial z}{\partial \alpha_1} \\ \dfrac{\partial x}{\partial \alpha_2} & \dfrac{\partial y}{\partial \alpha_2} & \dfrac{\partial z}{\partial \alpha_2} \end{vmatrix}. \end{aligned} \tag{2.4.16}$$

Define

$$\left.\begin{aligned} L &= b_{11} \\ M &= b_{12} = b_{21} \\ N &= b_{22} \end{aligned}\right\}. \tag{2.4.17}$$

Substitute (2.4.17) into (2.4.10).

$$k = \frac{L(d\alpha_1)^2 + 2M d\alpha_1 d\alpha_2 + N(d\alpha_2)^2}{E(d\alpha_1)^2 + 2F d\alpha_1 d\alpha_2 + G(d\alpha_2)^2} . \tag{2.4.18}$$

We now have the curvature in terms of the first and second fundamental forms. The next step will be to maximize (2.4.18) to obtain the principal directions. Let $\alpha_2 = \alpha_2(\alpha_1)$ and $\lambda = d\alpha_2/d\alpha_1$, where λ is an unspecified parametric direction. From (2.4.18)

$$k = \frac{L + 2M\left(\frac{d\alpha_2}{d\alpha_1}\right) + N\left(\frac{d\alpha_2}{d\alpha_1}\right)^2}{E + 2F\left(\frac{d\alpha_2}{d\alpha_1}\right) + G\left(\frac{d\alpha_2}{d\alpha_1}\right)^2}$$

$$= \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2} . \tag{2.4.19}$$

To find the directions for which k is an extremum, take the derivative of (2.4.19) with respect to λ, and set this equal to zero.

$$\frac{dk}{d\lambda} = \frac{(2M + 2N\lambda)}{(E + 2F\lambda + G\lambda^2)} - \frac{(L + 2M\lambda + N\lambda^2)(2F + G\lambda)}{(E + 2F\lambda + G\lambda^2)^2}$$

$$= 0 \tag{2.4.20}$$

Substitute (2.4.19) into (2.4.20).

$$\frac{dk}{d\lambda} = \frac{(2M + 2N\lambda)}{(E + 2F\lambda + G\lambda^2)} - \frac{k(2F + 2G\lambda)}{(E + 2F\lambda + G\lambda^2)}$$

$$= 0 .$$

Since the denominator will never be zero,

$$2M + 2N\lambda - k(2F + 2G\lambda) = 0$$

$$k = \frac{M + N\lambda}{F + G\lambda} . \tag{2.4.21}$$

Write (2.4.19) as

$$k = \frac{(L + M\lambda) + \lambda(M + N\lambda)}{(E + F\lambda) + \lambda(F + G\lambda)}$$

$$k[(E + F\lambda) + \lambda(F + G\lambda)] = (L + M\lambda) + \lambda(M + N\lambda) . \tag{2.4.22}$$

Substitute (2.4.21) into (2.4.22).

$$k[(E + F\lambda) + (F + G\lambda)\lambda] = (L + M\lambda) + \lambda(F + G\lambda)k$$

$$k(E + F\lambda) = L + M\lambda$$

$$k = \frac{L + M\lambda}{E + F\lambda}. \tag{2.4.23}$$

Cross multiply (2.4.21) and (2.4.23) to form a quadratic in λ which will yield the principal directions.

$$(L + M\lambda)(F + G\lambda) = (M + N\lambda)(E + F\lambda)$$

$$LF + FM\lambda + GL\lambda + MG\lambda^2 = ME + MF\lambda + NE\lambda + NF\lambda^2$$

$$\lambda^2(MG - NF) + (LG - NE)\lambda + (LF - ME) = 0\,. \tag{2.4.24}$$

The solutions to (2.4.24) are

$$\begin{Bmatrix} \lambda_1 \\ \lambda_2 \end{Bmatrix} = \frac{-(LG - NE) \pm \sqrt{(LG - NE)^2 - 4(MG - MF)(LF - ME)}}{2(MG - MF)}. \tag{2.4.25}$$

Apply the theory of equations for a quadratic to (2.4.24).

$$\lambda_1 + \lambda_2 = -\frac{(LG - NE)}{(MG - NF)} \tag{2.4.26}$$

$$\lambda_1\lambda_2 = \frac{(LF - ME)}{(MG - NF)}. \tag{2.4.27}$$

The two principal directions will now be shown to be orthogonal. Let

$$\lambda_1 = \left(\frac{d\alpha_2}{d\alpha_1}\right) \tag{2.4.28}$$

$$\lambda_2 = \left(\frac{\delta\alpha_2}{\delta\alpha_1}\right). \tag{2.4.29}$$

Let θ be the angle between these two directions, and let $d\mathbf{r}$ and $\delta\mathbf{r}$ be infinitesimal vectors along λ_1 and λ_2. The cosine of the angle between the vectors is

$$\cos\theta = \frac{d\mathbf{r}}{|d\mathbf{r}|} \cdot \frac{\delta\mathbf{r}}{|\delta\mathbf{r}|}$$

$$= \frac{d\mathbf{r}}{ds} \cdot \frac{\delta\mathbf{r}}{\delta s}. \tag{2.4.30}$$

The total differential can be developed

$$dr = \frac{\partial r}{\partial \alpha_1} d\alpha_1 + \frac{\partial r}{\partial \alpha_2} d\alpha_2 \qquad (2.4.31)$$

$$\delta r = \frac{\partial r}{\partial \alpha_1} \delta\alpha_1 + \frac{\partial r}{\partial \alpha_2} \delta\alpha_2 . \qquad (2.4.32)$$

Substitute (2.4.31) and (2.4.32) into (2.4.30).

$$\cos\theta = \left[\left(\frac{\partial r}{\partial \alpha_1} \cdot \frac{\partial r}{\partial \alpha_1}\right) d\alpha_1 \delta\alpha_1 + \left(\frac{\partial r}{\partial \alpha_1} \cdot \frac{\partial r}{\partial \alpha_2}\right) d\alpha_1 \delta\alpha_2 \right.$$

$$\left. + \left(\frac{\partial r}{\partial \alpha_2} \cdot \frac{\partial r}{\partial \alpha_1}\right) d\alpha_2 \delta\alpha_1 + \left(\frac{\partial r}{\partial \alpha_2} \cdot \frac{\partial r}{\partial \alpha_2}\right) d\alpha_2 \delta\alpha_2 \right] \times \frac{1}{ds\delta s} \qquad (2.4.33)$$

Substitute (2.3.2) into (2.4.33).

$$\cos\theta = \frac{1}{ds\delta s} \left[E d\alpha_1 \delta\alpha_1 + F(d\alpha_1 \delta\alpha_2 + d\alpha_2 \delta\alpha_1) + G d\alpha_2 \delta\alpha_2\right]$$

$$\frac{\cos\theta}{d\alpha_1 \delta\alpha_1} = \frac{1}{ds\delta s} \left[E + F\left(\frac{\delta\alpha_2}{\delta\alpha_1} + \frac{d\alpha_2}{d\alpha_1}\right) + G\left(\frac{\delta\alpha_2}{\delta\alpha_1}\right)\left(\frac{d\alpha_2}{d\alpha_1}\right)\right]. \qquad (2.4.34)$$

Substitute (2.4.28) and (2.4.29) into (2.4.34).

$$\frac{\cos\theta}{\delta\alpha_1 d\alpha_1} = \frac{1}{ds\delta s} \left[E + F(\lambda_1 + \lambda_2) + G(\lambda_1 \lambda_2)\right] . \qquad (2.4.35)$$

Substitute (2.4.26) and (2.4.27) into (2.4.35).

$$\frac{\cos\theta}{\delta\alpha_1 d\alpha_1} = \frac{1}{ds\delta s}\left[E + F\,\frac{(LG - NE)}{MG - NF} + G\,\frac{(LF - ME)}{MG - NF}\right]$$

$$= \frac{1}{ds\delta s}\left[\frac{EMG - ENF - FLG + ENF + FLG - EMG}{MG - NF}\right] = 0 .$$

Thus, $\theta = 90°$, and the principal directions are orthogonal. Since the principal directions are orthogonal we can choose the parametric curves to coincide with the directions of principal curvature. This provides additional simplification.

The equations of the lines of curvature are

$$\lambda_1 = \lambda_2 = 0 . \qquad (2.4.36)$$

If the lines of principal curvature coincide with the parametric lines, from (2.4.36), and (2.4.26)

$$\left.\begin{aligned} MG - NF &= 0 \\ LF - ME &= 0 \end{aligned}\right\} . \tag{2.4.37}$$

From orthogonality, $F = 0$. This means from (2.4.37)

$$\begin{aligned} MG &= 0 \\ ME &= 0 . \end{aligned} \tag{2.4.38}$$

For an actual surface, neither E nor G can be zero. Thus from (2.4.38), $M = 0$.

Substitute $F = M = 0$ into (2.4.21) and (2.4.23)

$$k_1 = \frac{L}{E} \tag{2.4.39}$$

$$k_2 = \frac{N}{G} . \tag{2.4.40}$$

Equations (2.4.39) and (2.4.40) give the means of obtaining the principal curvature of a surface from the first and second fundamental quantities. This technique will be applied to the surfaces of interest to map projection.

2.5 Surfaces of Revolution [10]

Surfaces of revolution are formed when a space curve is rotated about an axis. The two parameters needed to define a position on the surface of revolution will be z, and λ. Figure 2.5.1 gives the geometry for the development.

Let $R_0 = R_0(z)$. The position of the point P is

$$\mathbf{r} = R_0 \cos\lambda \hat{i} + R_0 \sin\lambda \hat{j} + z\hat{k} . \tag{2.5.1}$$

From (2.2.2) and (2.2.3)

$$\mathbf{a}_1 = \frac{\partial R_0}{\partial z} \cos\lambda \hat{i} + \frac{\partial R_0}{\partial z} \sin\lambda \hat{j} + \hat{k} \tag{2.5.2}$$

$$\mathbf{a}_2 = -R_0 \sin\lambda \hat{i} + R_0 \cos\lambda \hat{j} . \tag{2.5.3}$$

From (2.3.10), the normal to the surface is

$$\hat{n} = -\frac{\mathbf{a}_1 \times \mathbf{a}_2}{\sqrt{EG - F^2}} . \tag{2.5.4}$$

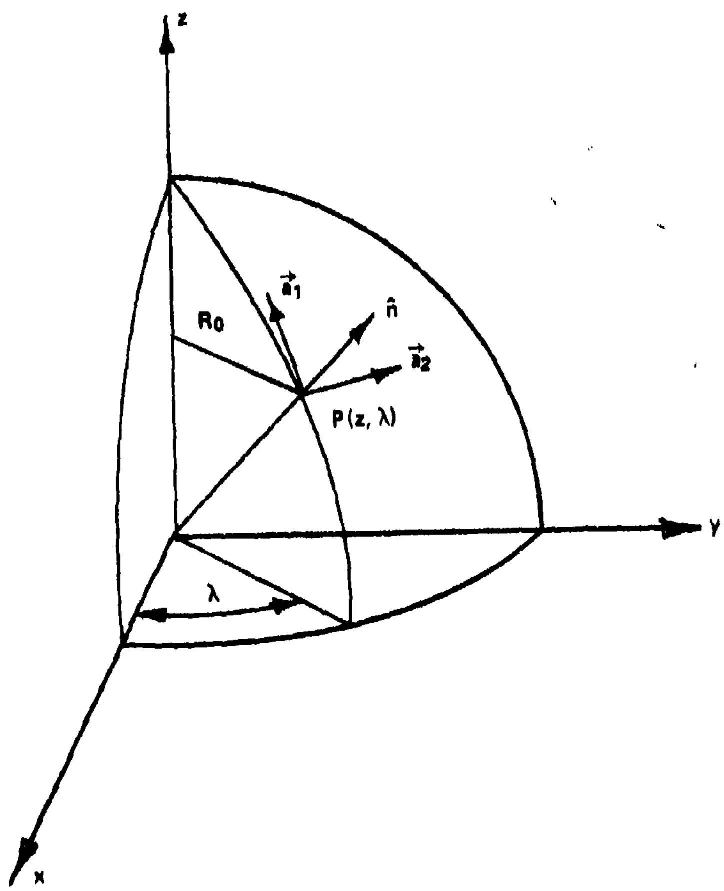


Figure 2.5.1. Geometry for a surface of revolution

The first fundamental quantities are, from (2.3.2)

$$E = \left(\frac{\partial R_0}{\partial z}\right)^2 \cos^2\lambda + \left(\frac{\partial R_0}{\partial z}\right)^2 \sin^2\lambda + 1 = 1 + \left(\frac{\partial R_0}{\partial z}\right)^2 \tag{2.5.5}$$

$$F = -\frac{\partial R_0}{\partial z} \cos\lambda\, R_0 \sin\lambda + \frac{\partial R_0}{\partial z} \sin\lambda\, R_0 \cos\lambda = 0 \tag{2.5.6}$$

$$G = R_0^2 \sin^2\lambda + R_0^2 \cos^2\lambda$$

$$= R_0^2 \tag{2.5.7}$$

$$\mathbf{a}_1 \times \mathbf{a}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial R_0}{\partial z} \cos\lambda & \frac{\partial R_0}{\partial z} \sin\lambda & 1 \\ -R_0 \sin\lambda & R_0 \cos\lambda & 0 \end{vmatrix}$$

$$= -R_0 \cos\lambda \hat{\imath} - R_0 \sin\lambda \hat{\jmath}$$

$$+\left(R_0 \frac{\partial R_0}{\partial z} \cos^2\lambda + R_0 \frac{\partial R_0}{\partial z} \sin^2\lambda\right) \hat{k}$$

$$= -R_0\left(\cos\lambda\hat{\imath} + \sin\lambda\hat{\jmath} - \frac{\partial R_0}{\partial z} \hat{k}\right). \tag{2.5.8}$$

Substitute (2.5.5), (2.5.6), (2.5.7), and (2.5.8) into (2.5.4).

$$\hat{n} = -\frac{R_0\left(\cos\lambda\hat{\imath} + \sin\lambda\hat{\jmath} - \frac{\partial R_0}{\partial z} \hat{k}\right)}{R_0 \sqrt{1 + \left(\frac{\partial R_0}{\partial z}\right)^2}}$$

$$= -\frac{\cos\lambda\hat{\imath} + \sin\lambda\hat{\jmath} - \frac{\partial R_0}{\partial z} \hat{k}}{\sqrt{1 + \left(\frac{\partial R_0}{\partial z}\right)^2}}. \tag{2.5.9}$$

The second fundamental quantities are given by (2.4.16), (2.4.17), and (2.5.9).

$$L = -\frac{\partial^2 r}{\partial z^2} \cdot \hat{n}$$

$$= -\left(\frac{\partial^2 R_0}{\partial z^2} \cos\lambda\hat{\imath} + \frac{\partial^2 R_0}{\partial z^2} \sin\lambda\hat{\jmath}\right)$$

$$\cdot \frac{\left(\cos\lambda\hat{\imath} + \sin\lambda\hat{\jmath} - \frac{\partial R_0}{\partial z}\hat{k}\right)}{\sqrt{1 + \left(\frac{\partial R_0}{\partial z}\right)^2}}$$

$$= -\frac{\frac{\partial^2 R_0}{\partial z^2} \cos^2\lambda + \frac{\partial^2 R_0}{\partial z^2} \sin^2\lambda}{\sqrt{1 + \left(\frac{\partial R_0}{\partial z}\right)^2}}$$

$$= -\frac{\frac{\partial^2 R_0}{\partial z^2}}{\sqrt{1 + \left(\frac{\partial R_0}{\partial z}\right)^2}}. \qquad (2.5.10)$$

$$M = -\frac{\partial^2 r}{\partial z \partial\lambda} \cdot \hat{n}$$

$$= -\left(-\frac{\partial R_0}{\partial z} \sin\lambda\hat{\imath} + \frac{\partial R_0}{\partial z} \cos\lambda\hat{\jmath}\right)$$

$$\cdot \frac{\cos\lambda\hat{\imath} + \sin\lambda\hat{\jmath} - \left(\frac{\partial R_0}{\partial z}\right)\hat{k}}{\sqrt{1 + \left(\frac{\partial R_0}{\partial z}\right)^2}} \qquad (2.5.11)$$

Continued

$$= -\frac{\left(-\dfrac{\partial R_0}{\partial z}\sin\lambda\cos\lambda + \dfrac{\partial R_0}{\partial z}\sin\lambda\cos\lambda\right)}{\sqrt{1 + \left(\dfrac{\partial R_0}{\partial z}\right)^2}} = 0 \qquad (2.5.11)$$

$$N = -\frac{\partial^2 \mathbf{r}}{\partial \lambda^2} \cdot \hat{n}$$

$$= -(-R_0 \cos\lambda \hat{\imath} - R_0 \sin\lambda \hat{\jmath})$$

$$\cdot \frac{\left(\cos\lambda\hat{\imath} + \sin\lambda\hat{\jmath} - \dfrac{\partial R_0}{\partial z}\hat{k}\right)}{\sqrt{1 + \left(\dfrac{\partial R_0}{\partial z}\right)^2}}$$

$$= \frac{R_0 \cos^2\lambda + R_0 \sin^2\lambda}{\sqrt{1 + \left(\dfrac{\partial R_0}{\partial z}\right)^2}}$$

$$N = \frac{R_0}{\sqrt{1 + \left(\dfrac{\partial R_0}{\partial z}\right)^2}}. \qquad (2.5.12)$$

To obtain the curvature, substitute (2.5.5), (2.5.7), (2.5.10) and (2.5.12) into (2.4.39) and (2.4.40).

$$k_1 = -\frac{\dfrac{\dfrac{\partial^2 R_0}{\partial z^2}}{\sqrt{1 + \left(\dfrac{\partial R_0}{\partial z}\right)^2}}}{1 + \left(\dfrac{\partial R_0}{\partial z}\right)^2}$$

$$= -\frac{\dfrac{\partial^2 R_0}{\partial z^2}}{\left[1 + \left(\dfrac{\partial R_0}{\partial z}\right)^2\right]^{3/2}}. \qquad (2.5.13)$$

Note that this is comparable to (2.1.34).

$$k_2 = \frac{\dfrac{R_0}{\sqrt{1 + \left(\dfrac{\partial R_0}{\partial z}\right)^2}}}{R_0^2}$$

$$= \frac{1}{R_0\sqrt{1 + \left(\dfrac{\partial R_0}{\partial z}\right)^2}} . \qquad (2.5.14)$$

From Figure 2.5.2, we can find the relations in the meridian plane.

$$\cos\phi = \frac{\mathbf{a}_1}{\sqrt{E}} \cdot \hat{k} . \qquad (2.5.15)$$

Substitute (2.5.2) and (2.5.5) into (2.5.15)

$$\cos\phi = \frac{\left(\dfrac{\partial R_0}{\partial z}\cos\lambda\hat{\imath} + \dfrac{\partial R_0}{\partial z}\sin\lambda\hat{\jmath} + \hat{k}\right)}{\sqrt{1 + \left(\dfrac{\partial R_0}{\partial z}\right)^2}} \cdot \hat{k}$$

$$= \frac{1}{\sqrt{1 + \left(\dfrac{\partial R_0}{\partial z}\right)^2}} . \qquad (2.5.16)$$

From the figure

$$R_2 = \frac{R_0}{\cos\phi} . \qquad (2.5.17)$$

Eliminating $\cos\phi$ between (2.5.16) and (2.5.17)

$$R_2 = R_0\sqrt{1 + \left(\frac{\partial R_0}{\partial z}\right)^2} \qquad (2.5.18)$$

R_2 is the second radius of curvature, and is the inverse of (2.5.14).

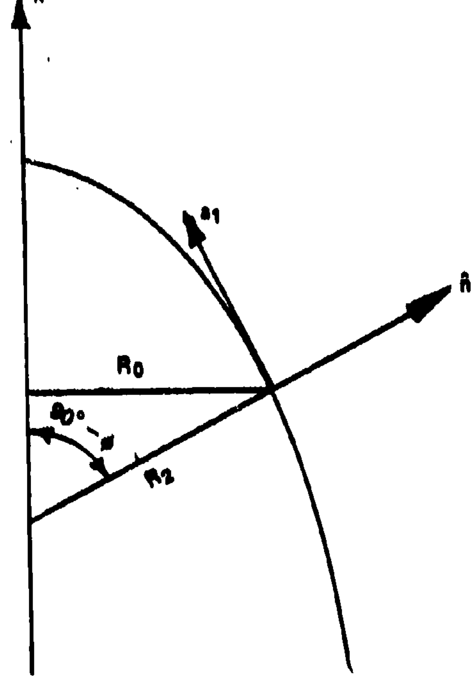


Figure 2.5.2. Geometry of the meridian curve

2.6 Developable Surfaces [10]

It was mentioned in Chapter 1 that there are two types of surfaces of interest to map projections: developable and non-developable. One way to make the distinction between the two is to consider the principal radii of curvature. Non-developable surfaces have two finite radii of curvature. Developable surfaces have one finite and one infinite radius of curvature. This section will expand on the differences between the two types of surfaces. This section will consider the two developable of surfaces of interest to mapping, the cone and cylinder, and the non-developable surface the sphere.

The surfaces which are envelopes of one-parameter families of planes are called developable surfaces. Every cone or cylinder is an envelope of a one-parameter family of tangent planes. Moreover, every tangent plane has a contact with the surface along a straight line. Consequently, a developable surface is swept out by a family of rectilinear generators.

It will be necessary to consider the tangent planes for cones, cylinders, and spheres, and note their characteristics.

For the cone, consider arbitrary parameters u and v. Let the origin of the coordinate system be at the vertex of the cone. The parametric equation of the cone is

$$\mathbf{r} = v\mathbf{q}(u) . \tag{2.6.1}$$

From (2.2.2) and (2.2.3)

$$\mathbf{a}_1 = v\dot{\mathbf{q}}(u) \tag{2.6.2}$$

$$\mathbf{a}_2 = \mathbf{q}(u) . \tag{2.6.3}$$

Take the cross product of (2.6.2) and (2.6.3).

$$\mathbf{a}_1 \times \mathbf{a}_2 = v\dot{\mathbf{q}}(u) \times \mathbf{q}(u) . \tag{2.6.4}$$

If $\mathbf{q}(u)$ and $\dot{\mathbf{q}}(u)$ are not collinear, the point (u, v) is regular, and the tangent plane has the equation, after the substitution of (2.6.1) and (2.6.4)

$$[\mathbf{r} - v\mathbf{q}(u)] \cdot v\dot{\mathbf{q}}(u) \times \mathbf{q}(u) = 0$$

$$\mathbf{r} \cdot \dot{\mathbf{q}}(u) \times \mathbf{q}(u) = 0 . \tag{2.6.5}$$

Thus, (2.6.5) depends only on u, and the family of tangent planes is a one-parameter family.

For a cylinder with elements parallel to a constant vector $\mathbf{c}$, the parametric equation is

$$\mathbf{r} = \mathbf{q}(u) + v\mathbf{c} . \tag{2.6.6}$$

Applying (2.2.2) and (2.2.3) to (2.2.6),

$$\mathbf{a}_1 = \dot{\mathbf{q}}(u) \tag{2.6.7}$$

$$\mathbf{a}_2 = \mathbf{c} . \tag{2.6.8}$$

Taking the cross product of (2.6.7) and (2.6.8)

$$\mathbf{a}_1 \times \mathbf{a}_2 = \dot{\mathbf{q}}(u) \times \mathbf{c} . \tag{2.6.9}$$

From (2.6.6) and (2.6.9), the equation of the tangent plane is

$$[\mathbf{r} - \mathbf{q}(u) - v\mathbf{c}] \cdot \dot{\mathbf{q}}(u) \times \mathbf{c} = 0$$

$$\mathbf{r} \cdot \dot{\mathbf{q}}(u) \times \mathbf{c} = \mathbf{q}(u) \cdot \dot{\mathbf{q}}(u) \times \mathbf{c} . \tag{2.6.10}$$

Again, (2.6.10) depends only on the parameter u.

A different situation occurs when a non-developable surface such as the sphere is considered. Let the two parameters be ϕ and λ. The equation of the surface is

$$\mathbf{r} = a(\cos\lambda \cos\phi \hat{\imath} + \sin\lambda \cos\phi \hat{\jmath} + \sin\phi \hat{k}) . \tag{2.6.11}$$

Using (2.2.2) and (2.2.3),

$$\mathbf{a}_1 = a(-\cos\lambda \sin\phi \hat{\imath} - \sin\lambda \sin\phi \hat{\jmath} + \cos\phi \hat{k}) . \tag{2.6.12}$$

$$\mathbf{a}_2 = a(-\sin\lambda \cos\phi \hat{\imath} + \cos\lambda \sin\phi \hat{\jmath}) \tag{2.6.13}$$

Taking the cross product of (2.6.12) and (2.6.13)

$$\begin{aligned}
\mathbf{a}_1 \times \mathbf{a}_2 &= a^2 \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\cos\lambda \sin\phi & -\sin\lambda \cos\phi & \cos\phi \\ -\sin\lambda \cos\phi & \cos\lambda \cos\phi & 0 \end{vmatrix} \\
&= a^2 [-\cos\lambda \cos^2\phi \hat{\imath} - \sin\lambda \cos^2\phi \hat{\jmath} \\
&\quad -\hat{k}(\cos^2\lambda \sin\phi \cos\phi + \sin^2\lambda \sin\phi \cos\phi)] \\
&= -a^2 [\cos\lambda \cos^2\phi \hat{\imath} + \sin\lambda \cos^2\phi \hat{\jmath} + \cos\phi \sin\phi \hat{k}] .
\end{aligned} \tag{2.6.14}$$

From (2.6.11) and (2.6.14), the tangent plane to the sphere has the equation

$$\begin{aligned}
&[\mathbf{r} - a(\cos\lambda \cos\phi \hat{\imath} + \sin\lambda \cos\phi \hat{\jmath} + \sin\phi \hat{k})] \\
&\cdot [-a^2 (\cos\lambda \cos^2\phi \hat{\imath} + \sin\lambda \cos^2\phi \hat{\jmath} + \cos\phi \sin\phi \hat{k}] = 0 .
\end{aligned} \tag{2.6.15}$$

Equation (2.6.15) depends on two parameters, and thus, the sphere is a non-developable surface.

2.7 Transformation Matrix [20], [19]

A transformation matrix will be derived which will permit the transformation from positions on the earth to places on the map. This will entail relating the fundamental quantities of the earth and the plotting surfaces by means of a Jacobian determinant.

Consider the earth surface, with parametric curves on it defined by ϕ and λ. The fundamental quantities will be defined as e, f, and g. The coordinates of point P on the earth, as in Figure 2.7.1, are given functionally as

$$\left.\begin{aligned} x &= x(\phi, \lambda) \\ y &= y(\phi, \lambda) \\ z &= z(\phi, \lambda) \end{aligned}\right\} . \tag{2.7.1}$$

Consider next an arbitrary projection surface, with parametric curves defined by the parameters u and v, with the fundamental quantities E', F', and G'. The position of the point P' on the plotting surface in Figure 2.7.2 is given functionally by

$$\left.\begin{aligned} X &= X(u, v) \\ Y &= Y(u, v) \\ Z &= Z(u, v) \end{aligned}\right\} . \tag{2.7.2}$$

The parametric curves on the earth are related to those on the projection surface

$$\begin{aligned} u &= u(\phi, \lambda) \\ v &= v(\phi, \lambda) . \end{aligned} \tag{2.7.3}$$

For the earth, and for any plotting surface, only two conditions are to be satisfied. The projection must be (1) unique, and (2) reversible. A point on the earth must correspond to only one point on the map, and vice versa. This requires that

$$\left.\begin{aligned} \phi &= \phi(u, v) \\ \lambda &= \lambda(u, v) \end{aligned}\right\} . \tag{2.7.4}$$

Substituting (2.7.3) into (2.7.2)

$$\begin{aligned} X &= X[u(\phi, \lambda), v(\phi, \lambda)] \\ Y &= Y[u(\phi, \lambda), v(\phi, \lambda)] \\ Z &= Z[u(\phi, \lambda), v(\phi, \lambda)] . \end{aligned} \tag{2.7.5}$$

In this form, the surface will have the fundamental quantities E, F, and G.

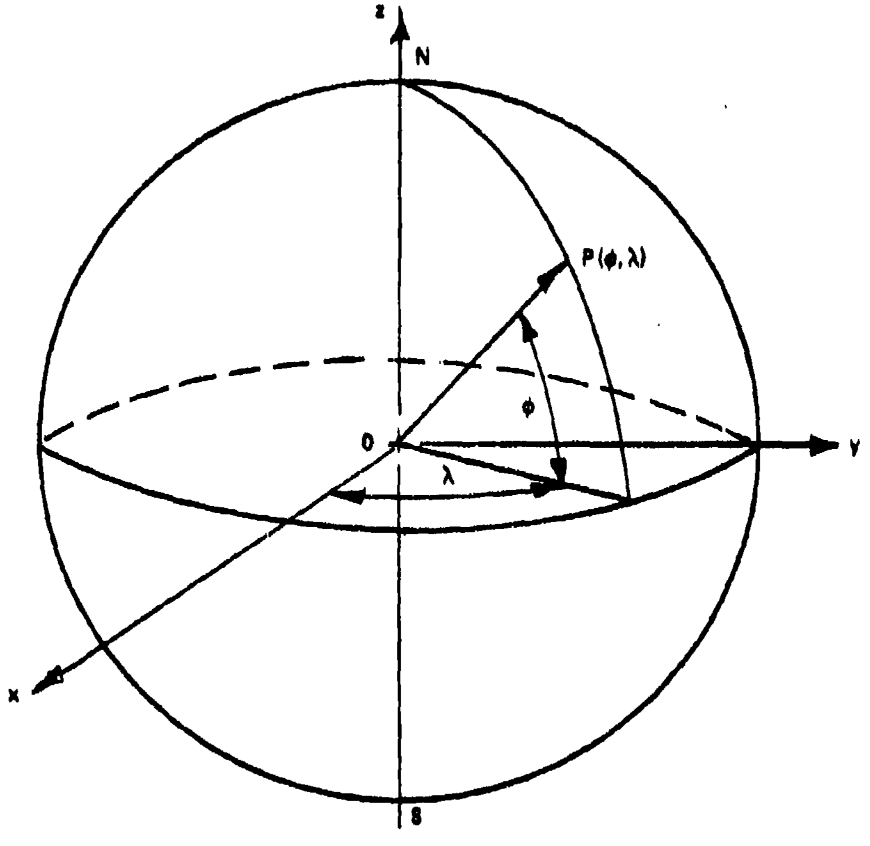


Figure 2.7.1. Parametric representation of point P on the earth

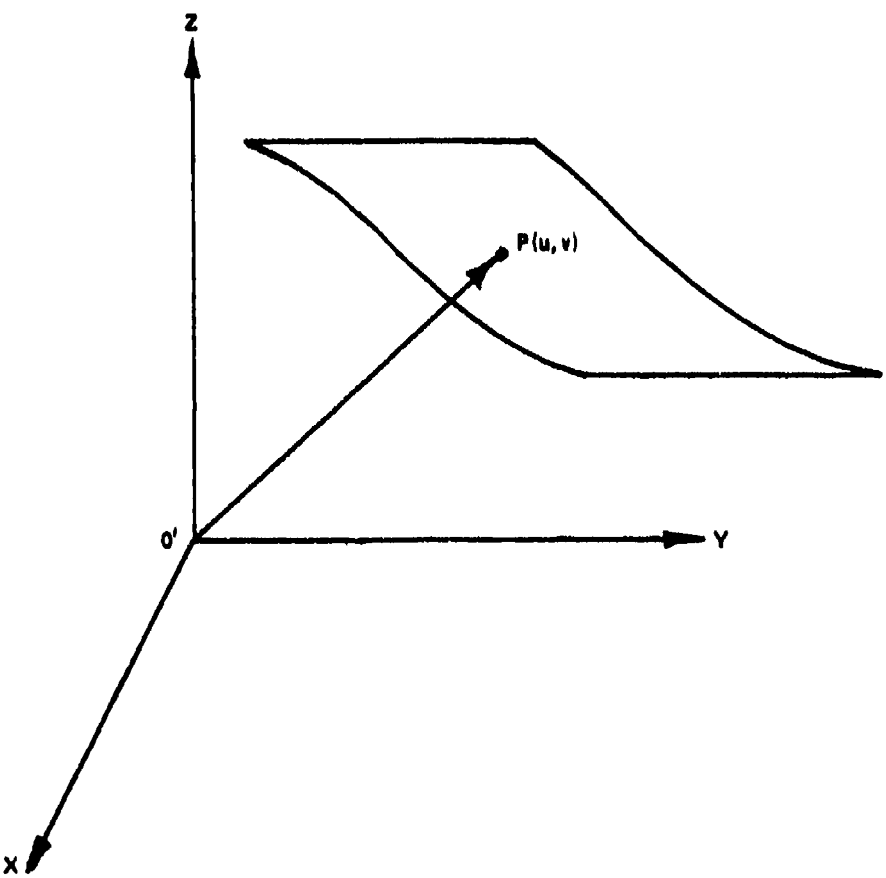


Figure 2.7.2. Parametric representation of point P' on the projection surface

Differentiate (2.7.5) with respect to ϕ, and λ.

$$
\left.\begin{aligned}
\frac{\partial X}{\partial \phi} &= \frac{\partial X}{\partial u}\frac{\partial u}{\partial \phi} + \frac{\partial X}{\partial v}\frac{\partial u}{\partial \phi} \\
\frac{\partial X}{\partial \lambda} &= \frac{\partial X}{\partial u}\frac{\partial u}{\partial \lambda} + \frac{\partial X}{\partial v}\frac{\partial v}{\partial \lambda} \\
\frac{\partial Y}{\partial \phi} &= \frac{\partial Y}{\partial u}\frac{\partial u}{\partial \phi} + \frac{\partial Y}{\partial v}\frac{\partial v}{\partial \phi} \\
\frac{\partial Y}{\partial \lambda} &= \frac{\partial Y}{\partial u}\frac{\partial u}{\partial \lambda} + \frac{\partial Y}{\partial v}\frac{\partial v}{\partial \lambda} \\
\frac{\partial Z}{\partial \phi} &= \frac{\partial Z}{\partial u}\frac{\partial u}{\partial \phi} + \frac{\partial Z}{\partial v}\frac{\partial v}{\partial \phi} \\
\frac{\partial Z}{\partial \lambda} &= \frac{\partial Z}{\partial u}\frac{\partial u}{\partial \lambda} + \frac{\partial Z}{\partial v}\frac{\partial v}{\partial \lambda}
\end{aligned}\right\} . \qquad (2.7.6)
$$

From Section 2.3

$$
\begin{aligned}
E &= \left(\frac{\partial X}{\partial \phi}\right)^2 + \left(\frac{\partial Y}{\partial \phi}\right)^2 + \left(\frac{\partial Z}{\partial \phi}\right)^2 \\
F &= \frac{\partial X}{\partial \phi}\frac{\partial X}{\partial \lambda} + \frac{\partial Y}{\partial \phi}\frac{\partial Y}{\partial \lambda} + \frac{\partial Z}{\partial \phi}\frac{\partial Z}{\partial \lambda} \\
G &= \left(\frac{\partial X}{\partial \lambda}\right)^2 + \left(\frac{\partial Y}{\partial \lambda}\right)^2 + \left(\frac{\partial Z}{\partial \lambda}\right)^2 . \qquad (2.7.7)
\end{aligned}
$$

Substitute (2.7.6) into (2.7.7)

$$
\begin{aligned}
E = &\left(\frac{\partial X}{\partial u}\frac{\partial u}{\partial \phi}\right)^2 + 2\,\frac{\partial X}{\partial u}\frac{\partial u}{\partial \phi}\frac{\partial X}{\partial v}\frac{\partial v}{\partial \phi} + \left(\frac{\partial X}{\partial v}\frac{\partial v}{\partial \phi}\right)^2 \\
&+\left(\frac{\partial Y}{\partial u}\frac{\partial u}{\partial \phi}\right)^2 + 2\,\frac{\partial Y}{\partial u}\frac{\partial u}{\partial \phi}\frac{\partial X}{\partial v}\frac{\partial v}{\partial \phi} + \left(\frac{\partial Y}{\partial v}\frac{\partial v}{\partial \phi}\right)^2 \\
&+\left(\frac{\partial Z}{\partial u}\frac{\partial u}{\partial \phi}\right)^2 + 2\,\frac{\partial Z}{\partial u}\frac{\partial u}{\partial \phi}\frac{\partial Z}{\partial v}\frac{\partial v}{\partial \phi} + \left(\frac{\partial Z}{\partial v}\frac{\partial v}{\partial \phi}\right)^2 \qquad (2.7.8)
\end{aligned}
$$

Continued

$$G = \left(\frac{\partial X}{\partial u}\frac{\partial u}{\partial \lambda}\right)^2 + 2\,\frac{\partial X}{\partial u}\frac{\partial u}{\partial \lambda}\frac{\partial X}{\partial v}\frac{\partial v}{\partial \lambda} + \left(\frac{\partial X}{\partial v}\frac{\partial v}{\partial \lambda}\right)^2$$

$$+\left(\frac{\partial Y}{\partial u}\frac{\partial u}{\partial \lambda}\right)^2 + 2\,\frac{\partial Y}{\partial u}\frac{\partial u}{\partial \lambda}\frac{\partial Y}{\partial v}\frac{\partial v}{\partial \lambda} + \left(\frac{\partial Y}{\partial v}\frac{\partial v}{\partial \lambda}\right)^2$$

$$+\left(\frac{\partial Z}{\partial u}\frac{\partial u}{\partial \lambda}\right)^2 + 2\,\frac{\partial Z}{\partial u}\frac{\partial u}{\partial \lambda}\frac{\partial Z}{\partial v}\frac{\partial v}{\partial \lambda} + \left(\frac{\partial Z}{\partial v}\frac{\partial v}{\partial \lambda}\right)^2$$

$$F = \left(\frac{\partial X}{\partial u}\right)^2\frac{\partial u}{\partial \phi}\frac{\partial u}{\partial \lambda} + \frac{\partial X}{\partial u}\frac{\partial X}{\partial v}\left(\frac{\partial u}{\partial \phi}\frac{\partial v}{\partial \lambda} + \frac{\partial u}{\partial \lambda}\frac{\partial v}{\partial \phi}\right) + \left(\frac{\partial X}{\partial v}\right)^2\frac{\partial v}{\partial \phi}\frac{\partial v}{\partial \lambda}$$

$$+\left(\frac{\partial Y}{\partial u}\right)^2\frac{\partial u}{\partial \phi}\frac{\partial u}{\partial \lambda} + \frac{\partial Y}{\partial u}\frac{\partial Y}{\partial v}\left(\frac{\partial u}{\partial \phi}\frac{\partial v}{\partial \lambda} + \frac{\partial u}{\partial \lambda}\frac{\partial v}{\partial \phi}\right) + \left(\frac{\partial Y}{\partial v}\right)^2\frac{\partial v}{\partial \phi}\frac{\partial v}{\partial \lambda}$$

$$+\left(\frac{\partial Z}{\partial u}\right)^2\frac{\partial u}{\partial \phi}\frac{\partial u}{\partial \lambda} + \frac{\partial Z}{\partial u}\frac{\partial Z}{\partial v}\left(\frac{\partial u}{\partial \phi}\frac{\partial v}{\partial \lambda} + \frac{\partial u}{\partial \lambda}\frac{\partial v}{\partial \phi}\right) + \left(\frac{\partial Z}{\partial v}\right)^2\frac{\partial v}{\partial \phi}\frac{\partial v}{\partial \lambda}. \qquad (2.7.8)$$

Also, from Section 2.3

$$E' = \left(\frac{\partial X}{\partial u}\right)^2 + \left(\frac{\partial Y}{\partial u}\right)^2 + \left(\frac{\partial Z}{\partial u}\right)^2$$

$$F' = \frac{\partial X}{\partial u}\frac{\partial X}{\partial v} + \frac{\partial Y}{\partial u}\frac{\partial Y}{\partial v} + \frac{\partial Z}{\partial u}\frac{\partial Z}{\partial v}$$

$$G' = \left(\frac{\partial X}{\partial v}\right)^2 + \left(\frac{\partial Y}{\partial v}\right)^2 + \left(\frac{\partial Z}{\partial v}\right)^2. \qquad (2.7.9)$$

Substitute (2.7.9) into (2.7.8).

$$E = \left(\frac{\partial u}{\partial \phi}\right)^2 E' + 2\,\frac{\partial u}{\partial \phi}\frac{\partial v}{\partial \phi} F' + \left(\frac{\partial v}{\partial \phi}\right)^2 G'$$

$$F = \left(\frac{\partial u}{\partial \phi}\frac{\partial u}{\partial \lambda}\right) E' + \left(\frac{\partial u}{\partial \phi}\frac{\partial v}{\partial \lambda} + \frac{\partial u}{\partial \lambda}\frac{\partial v}{\partial \phi}\right) F' + \left(\frac{\partial v}{\partial \phi}\frac{\partial v}{\partial \lambda}\right) G'$$

$$G = \left(\frac{\partial u}{\partial \lambda}\right)^2 E' + 2\,\frac{\partial u}{\partial \lambda}\frac{\partial v}{\partial \lambda} F' + \left(\frac{\partial v}{\partial \lambda}\right)^2 G'. \qquad (2.7.10)$$

Equation (2.9.10) may be written in matrix notation as

$$\begin{bmatrix} E \\ F \\ G \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial u}{\partial \phi}\right)^2 & 2\,\frac{\partial u}{\partial \phi}\,\frac{\partial v}{\partial \phi} & \left(\frac{\partial v}{\partial \phi}\right)^2 \\ \frac{\partial u}{\partial \phi}\,\frac{\partial u}{\partial \lambda} & \frac{\partial u}{\partial \phi}\,\frac{\partial v}{\partial \lambda} + \frac{\partial u}{\partial \lambda}\,\frac{\partial v}{\partial \phi} & \frac{\partial v}{\partial \phi}\,\frac{\partial v}{\partial \lambda} \\ \left(\frac{\partial u}{\partial \lambda}\right)^2 & 2\,\frac{\partial u}{\partial \lambda}\,\frac{\partial v}{\partial \lambda} & \left(\frac{\partial v}{\partial \lambda}\right)^2 \end{bmatrix} \begin{bmatrix} E' \\ F' \\ G' \end{bmatrix} . \qquad (2.7.11)$$

The transformation matrix in (2.7.11) is the fundamental matrix for mapping transformations.

To facilitate the derivations of Chapters 4, 5, and 6, it will be useful to develop the term

$$H = EG - F^2 . \qquad (2.7.12)$$

From (2.7.10)

$$\begin{aligned} H = & \left[\left(\frac{\partial u}{\partial \phi}\right)^2 E' + 2\,\frac{\partial u}{\partial \phi}\,\frac{\partial v}{\partial \phi}\,F' + \left(\frac{\partial v}{\partial \phi}\right)^2 G'\right] \\ & \times\left[\left(\frac{\partial u}{\partial \lambda}\right)^2 E' + 2\,\frac{\partial u}{\partial \lambda}\,\frac{\partial v}{\partial \lambda}\,F' + \left(\frac{\partial v}{\partial \lambda}\right)^2 G'\right] \\ & -\left[\frac{\partial u}{\partial \phi}\,\frac{\partial u}{\partial \lambda}\,E' + \left(\frac{\partial u}{\partial \phi}\,\frac{\partial v}{\partial \lambda} + \frac{\partial u}{\partial \lambda}\,\frac{\partial v}{\partial \phi}\right) F' + \frac{\partial v}{\partial \phi}\,\frac{\partial v}{\partial \lambda}\,G'\right]^2 \end{aligned}$$

$$\begin{aligned} H = & (E')^2 \left(\frac{\partial u}{\partial \phi}\right)^2 \left(\frac{\partial u}{\partial \lambda}\right)^2 + 2\,\frac{\partial u}{\partial \phi}\,\frac{\partial v}{\partial \phi}\left(\frac{\partial u}{\partial \lambda}\right)^2 E'F' \\ & + \left(\frac{\partial v}{\partial \phi}\right)^2 \left(\frac{\partial u}{\partial \lambda}\right)^2 E'G' + 2\,\frac{\partial u}{\partial \lambda}\,\frac{\partial v}{\partial \lambda}\left(\frac{\partial u}{\partial \phi}\right)^2 E'F' \\ & +4\,\frac{\partial u}{\partial \phi}\,\frac{\partial v}{\partial \phi}\,\frac{\partial u}{\partial \lambda}\,\frac{\partial v}{\partial \lambda}\,(F')^2 + 2\,\frac{\partial u}{\partial \lambda}\,\frac{\partial v}{\partial \lambda}\left(\frac{\partial v}{\partial \phi}\right)^2 F'G' \\ & +\left(\frac{\partial u}{\partial \phi}\right)^2 \left(\frac{\partial v}{\partial \lambda}\right)^2 E'G' + 2\,\frac{\partial u}{\partial \phi}\,\frac{\partial v}{\partial \phi}\left(\frac{\partial v}{\partial \lambda}\right)^2 F'G' \\ & +\left(\frac{\partial u}{\partial \phi}\right)^2 \left(\frac{\partial v}{\partial \lambda}\right)^2 (G')^2 \end{aligned}$$

$$-\left(\frac{\partial u}{\partial \phi}\ \frac{\partial u}{\partial \phi}\right)^2 (E')^2 - 2\left(\frac{\partial u}{\partial \phi}\ \frac{\partial v}{\partial \lambda} + \frac{\partial u}{\partial \lambda}\ \frac{\partial v}{\partial \phi}\right) \frac{\partial u}{\partial \phi}\ \frac{\partial u}{\partial \lambda}\ E'F'$$

$$-2\ \frac{\partial u}{\partial \phi}\ \frac{\partial u}{\partial \lambda}\ \frac{\partial v}{\partial \phi}\ \frac{\partial v}{\partial \lambda}\ E'G' - \left(\frac{\partial u}{\partial \phi}\ \frac{\partial v}{\partial \lambda} + \frac{\partial u}{\partial x}\ \frac{\partial v}{\partial \phi}\right)^2 (F')^2$$

$$-\left(\frac{\partial v}{\partial \phi}\ \frac{\partial v}{\partial \lambda}\right)^2 (G')^2 - 2\left(\frac{\partial u}{\partial \phi}\ \frac{\partial v}{\partial \lambda} + \frac{\partial u}{\partial \lambda}\ \frac{\partial v}{\partial \phi}\right) \frac{\partial v}{\partial \phi}\ \frac{\partial v}{\partial \lambda}\ F'G'$$

$$= E'G'\left[\left(\frac{\partial u}{\partial \phi}\ \frac{\partial v}{\partial \lambda}\right)^2 + \left(\frac{\partial v}{\partial \phi}\ \frac{\partial u}{\partial \lambda}\right)^2 - 2\left(\frac{\partial u}{\partial \phi}\ \frac{\partial v}{\partial \lambda}\ \frac{\partial v}{\partial \phi}\ \frac{\partial u}{\partial \lambda}\right)\right]$$

$$+\left[4\ \frac{\partial u}{\partial \phi}\ \frac{\partial u}{\partial \lambda}\ \frac{\partial v}{\partial \phi}\ \frac{\partial v}{\partial \lambda} - \left(\frac{\partial u}{\partial \phi}\ \frac{\partial v}{\partial \lambda} + \frac{\partial u}{\partial \lambda}\ \frac{\partial v}{\partial \phi}\right)^2\right] (F')^2$$

$$= E'G'\left[\left(\frac{\partial u}{\partial \phi}\ \frac{\partial v}{\partial \lambda}\right)^2 + \left(\frac{\partial v}{\partial \phi}\ \frac{\partial u}{\partial \lambda}\right)^2 - 2\left(\frac{\partial u}{\partial \phi}\ \frac{\partial v}{\partial \lambda}\ \frac{\partial v}{\partial \phi}\ \frac{\partial u}{\partial \lambda}\right)\right]$$

$$+(F')^2\left[4\ \frac{\partial u}{\partial \phi}\ \frac{\partial u}{\partial \lambda}\ \frac{\partial v}{\partial \phi}\ \frac{\partial v}{\partial \phi} - \left(\frac{\partial u}{\partial \phi}\ \frac{\partial v}{\partial \lambda}\right)^2\right.$$

$$\left.-\left(\frac{\partial u}{\partial \lambda}\ \frac{\partial v}{\partial \phi}\right)^2 - 2\ \frac{\partial u}{\partial \phi}\ \frac{\partial u}{\partial \lambda}\ \frac{\partial v}{\partial \phi}\ \frac{\partial v}{\partial \lambda}\right]$$

$$= [E'G' - (F')^2] \times \left[\left(\frac{\partial u}{\partial \phi}\ \frac{\partial v}{\partial \lambda}\right)^2 + \left(\frac{\partial v}{\partial \phi}\ \frac{\partial u}{\partial \lambda}\right)^2\right.$$

$$\left.-2\left(\frac{\partial u}{\partial \phi}\ \frac{\partial v}{\partial \lambda}\ \frac{\partial v}{\partial \phi}\ \frac{\partial u}{\partial \lambda}\right)\right]$$

$$EG - F^2 = [E'G' - (F')^2]\left(\frac{\partial u}{\partial \phi}\ \frac{\partial v}{\partial \lambda} - \frac{\partial v}{\partial \phi}\ \frac{\partial u}{\partial \lambda}\right)^2$$

$$= \begin{vmatrix} E' & F' \\ F' & G' \end{vmatrix} \begin{vmatrix} \dfrac{\partial u}{\partial \phi} & \dfrac{\partial u}{\partial \lambda} \\[2ex] \dfrac{\partial v}{\partial \phi} & \dfrac{\partial v}{\partial \lambda} \end{vmatrix}^2 . \qquad (2.7.13)$$

The determinant

$$\begin{vmatrix} \dfrac{\partial u}{\partial \phi} & \dfrac{\partial u}{\partial \lambda} \\ \dfrac{\partial v}{\partial \phi} & \dfrac{\partial v}{\partial \lambda} \end{vmatrix}$$

is the Jacobean determinant of the transformation from (ϕ, λ) to (u, v).

2.8 Conditions for Equal Area and Conformal Projections [6], [10]

The first fundamental form and the fundamental quantities are used to define the conditions for equal area and conformal projections. This can be done in a general manner. For the conventional projections, each case has its own requirements, and no general relations can be defined.

An equal area map is one in which the areas of domains are preserved as they are transformed from the earth to the map. A theorem of differential geometry requires that a mapping from the earth to the plotting surface is locally equal area if, and only if,

$$eg - f^2 = EG - F^2 . \tag{2.8.1}$$

The relation (2.8.1) is substituted into (2.7.13) to obtain the equal area transformation. This is done in Chapter 4 to transform from the earth to the cylinder, plane and cone.

A mapping of the surface of the earth onto the plane, or a developable surface is called conformal (or isogonal) if it preserves the angle between intersecting curves on the surface. From a theorem of differential geometry, a mapping is called conformal if, and only if, the first fundamental forms of the earth and the mapping surface, in compatible coordinates, are proportional at every point. This requires that

$$\frac{E}{e} = \frac{F}{f} = \frac{G}{g} \tag{2.8.2}$$

in the symbols of the previous section.

The transformations of Chapter 5 will apply these relations between the earth, and the plane, cylinder, and cone.

2.9 Convergency of the Meridians [7]

As one goes pole-ward from the equator on the earth, the meridians converge, until, at the pole, all meridians intersect. This section will give an estimate of the degree of this convergency as a function of latitude. Both angular and linear convergency will be considered.

In Figure 2.9.1, ACN and BDN are two meridians separated by a longitude difference of $\Delta\lambda$. Let CD be an arc of the circle of latitude ϕ. Let the earth be considered as spherical.

From the figure,

$$CD = CO'\Delta\lambda \qquad (2.9.1)$$

$$DN = \frac{DO'}{\sin\phi}. \qquad (2.9.2)$$

Approximately, the angle of convergency is

$$\theta = \frac{CD}{DN'}. \qquad (2.9.3)$$

Substituting (2.9.1) and (2.9.2) into (2.9.3), and noting that $CO' = DO'$

$$\theta = \Delta\lambda \sin\phi\,. \qquad (2.9.4)$$

Let the distance between the meridians, measured along a parallel of latitude, be d, and let the radius of the earth be a. From the figure

$$\Delta\lambda = \frac{d}{a\cos\phi}. \qquad (2.9.5)$$

Substitute (2.9.5) into (2.9.4)

$$\theta = \frac{d\sin\phi}{a\cos\phi}$$

$$= \frac{d\tan\phi}{a}. \qquad (2.9.6)$$

The next step is to obtain the linear convergency. From Figure 2.9.2, let ℓ be the length of the meridian between two parallels ϕ_1 and ϕ_2. Let θ be the mean angular convergency at a mean latitude

$$\phi = \frac{\phi_1 + \phi_2}{2}. \qquad (2.9.7)$$

The mean distance, at the mean latitude, is d. Define the linear convergency of the two meridians to be c. Then, as an approximation,

$$\theta = c/\ell\,. \qquad (2.9.8)$$

Substitute (2.9.6) into (2.9.8).

$$c = \frac{d\cdot\ell\tan\phi}{a}. \qquad (2.9.9)$$

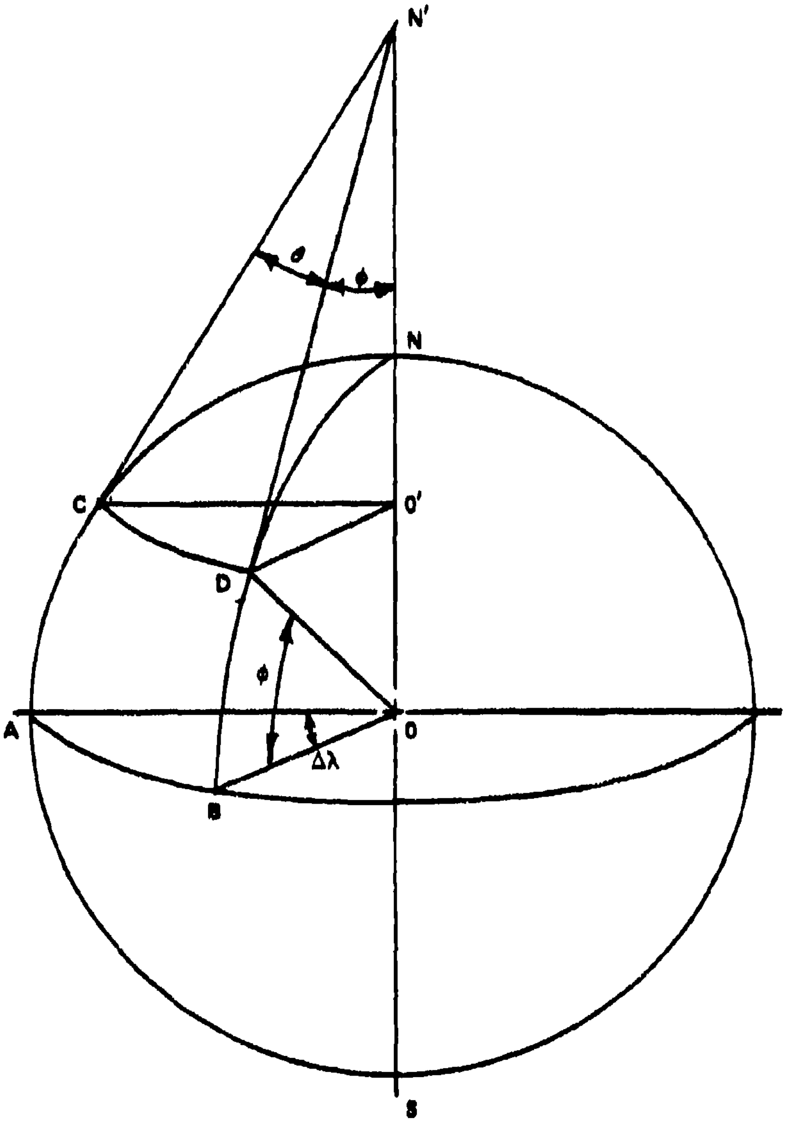


Figure 2.9.1. Geometry for angular convergence of the meridians

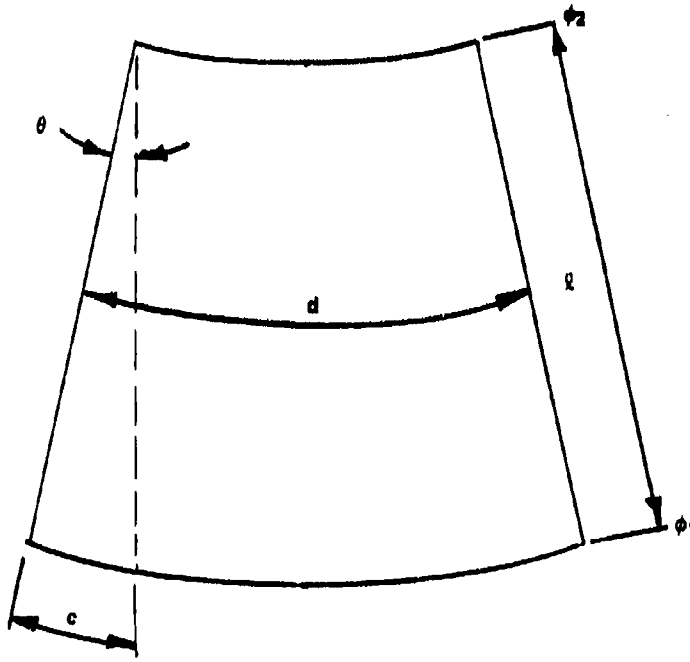


Figure 2.9.2. Geometry for linear convergence for the meridians

2.10 Rotation of the Coordinate System [20]

A rotation of the coordinate system can be defined to conveniently obtain oblique, transverse, and equatorial projections from polar projections. This can be conveniently done by applying formulas of spherical trigonometry. The spherical trigonometry approach is justified, since in Chapters 4 and 5, it will be shown that an intermediate transformation can be performed for the equal area and conformal projections which transforms from positions on the earth to the authalic or conformal sphere, respectively. Once this is done, the rotation formulas for the sphere can be applied directly. Also, the conventional projections of Chapter 6 are based on a spherical earth for the majority of their practical applications.

Figure 2.10.1 provides the basic geometry for the rotational transformation. Let Q be any arbitrary point with coordinates ϕ and λ on the earth. Let P be the pole of the auxiliary spherical coordinate system. In the standard equatorial coordinate system, P has the coordinates ϕ_p and λ_p. Let h be the latitude of Q in the auxiliary system, and α, the longitude in that same system. A reference meridian is chosen for the origin of measurement of α.

The intention is to derive the projection in the (h, α) system, and then transform to the (ϕ, λ) system for the plotting of the coordinates.

The relations between the angles of interest can be found from the spherical triangle PNQ.

From the law of cosines

$$\cos(90^\circ - \phi) = \cos(90^\circ - \phi_p)\cos(90^\circ - h) + \sin(90^\circ - \phi_p)\sin(90^\circ - h)\cos\alpha$$

$$\sin\phi = \sin\phi_p \sin h + \cos\phi_p \cos h \cos\alpha . \tag{2.10.1}$$

From the law of sines

$$\frac{\sin(\lambda - \lambda_p)}{\sin(90^\circ - h)} = \frac{\sin\alpha}{\sin(90^\circ - \phi)}$$

$$\sin(\lambda - \lambda_p) = \frac{\sin\alpha \cos h}{\cos\phi} . \tag{2.10.2}$$

Also, applying the four parts formula

$$\cos(90^\circ - \phi_p)\cos\alpha = \sin(90^\circ - \phi_p)\cot(90^\circ - h) - \sin\alpha\cot(\lambda - \lambda_p)$$

$$\sin\phi_p \cos\alpha = \cos\phi_p \tan h - \sin\alpha \cot(\lambda - \lambda_p)$$

$$\cot(\lambda - \lambda_p) = \frac{\cos\phi_p \tan h - \sin\phi_p \cos\alpha}{\sin\alpha} . \tag{2.10.3}$$

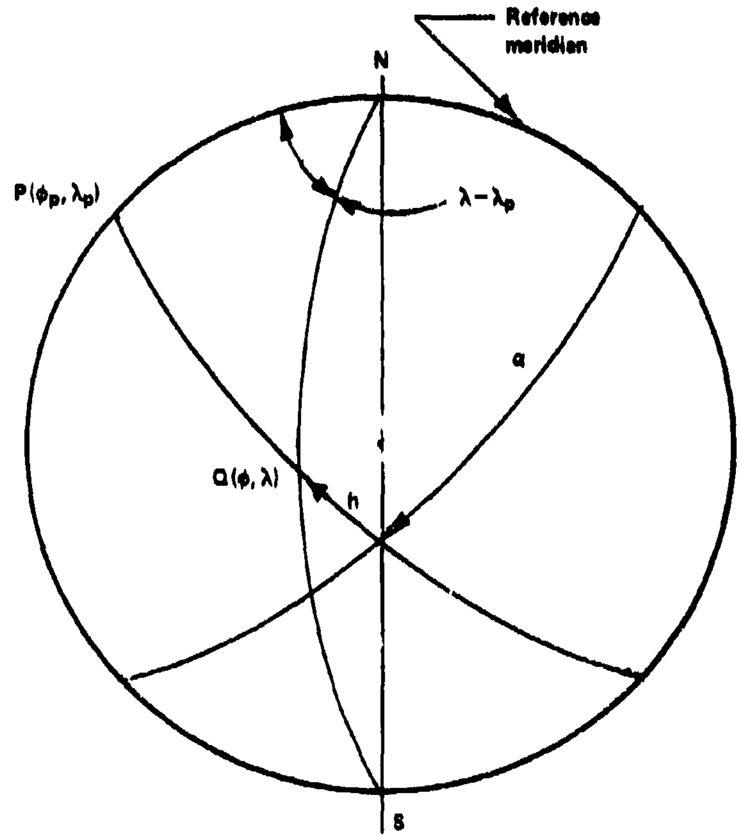


Figure 2.10.1. Geometry for the rotational transformation

The inverse relationships are also of use. From the law of cosines

$$\cos(90° - h) = \cos(90° - \phi)\cos(90° - \phi_p)$$

$$+ \sin(90° - \phi)\sin(90° - \phi_p)\cos(\lambda - \lambda_p)$$

$$\sin h = \sin\phi \sin\phi_p + \cos\phi \cos\phi_p \cos(\lambda - \lambda_p) . \qquad (2.10.4)$$

From the four parts formula

$$\cos(90° - \phi_p)\cos(\lambda - \lambda_p) = \sin(90° - \phi_p)\cot(90° - \phi)$$

$$- \sin(\lambda - \lambda_p)\cot\alpha$$

$$\sin\phi_p \cos(\lambda - \lambda_p) = \cos\phi_p \tan\phi - \sin(\lambda - \lambda_p)\cot\alpha$$

$$\sin(\lambda - \lambda_p)\cot\alpha = \cos\phi_p \tan\phi - \sin\phi_p \cos(\lambda - \lambda_p)$$

$$\tan\alpha = \frac{\sin(\lambda - \lambda_p)}{\cos\phi_p \tan\phi - \sin\phi_p \cos(\lambda - \lambda_p)} . \qquad (2.10.5)$$

A final useful equation is needed for unique quadrant determination. From a consideration of Figure 2.10.1

$$\cos\alpha \cos h = \sin\phi \cos\phi_p - \cos\phi \sin\phi_p \cos(\lambda - \lambda_p) . \qquad (2.10.6)$$

Having possession of equations (2.10.4), (2.10.5), and (2.10.6), we can accomplish any rotations necessary to form oblique, transverse, and equatorial projections from polar and regular projections. These will be required in some of the projections of Chapters 4, 5, and 6.

Chapter 3

FIGURE OF THE EARTH

The basic geometrical surface taken as the model of the earth will be an oblate spheroid generated by revolving an ellipse about its semi-minor axis. This chapter will be concerned with the geometry of the spheroid, and the reduction to the sphere.

The figure of the earth, as seen by the cartographer, is far less complex than that seen by the geodesist or the astronomer. For his basic surface, the cartographer may assume a single best spheroid, and project thereon positions from an undulating earth. Then, he is free to begin the process of projecting onto a map. The geodesist must consider a spheroid which is possibly best in one portion of the world, and other spheroids which are best in other portions of the world, and then strain to patch them together in a coordinated manner. The astronomer, dealing with the dynamical figure of the earth, must consider pear-shape and undulations to obtain solutions couched in tesseral harmonics.

The cartographer has only to deal with the geometrical figure of the earth, and can enjoy immense simplifications. To facilitate the transformations of Chapters 4, 5, and 6, it is necessary to consider the geometry of the ellipse and the spheroid. The coordinate system of the spheroid will be introduced. Angles and distances on the spheroid will be considered. Then, particular constants for the actual size and shape of the earth will be given.

Many of the projections in Chapters 4, 5, and 6 will be based on an intermediate transformation to a sphere. Thus, it is necessary to investigate the further simplifications in coordinates, angles and distances on a sphere.

Finally, the figure of the moon is developed. In the Space Age, maps of the moon have been required. It is fitting to give a standard reference spheroid for the moon.

3.1 Geometry of the Ellipse [16], [17]

The ellipse is the generating curve which produces the spheroid of revolution. The nomenclature of the ellipse is best described with reference to Figure 3.1.1.

The semi-major axis, a, is the length of the line AO, or the line OB. The semi-minor axis, b, is the length of the line DO, or the line CO. The equation for the ellipse, for a Cartesian coordinate system with origin at O, is

$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1 . \tag{3.1.1}$$

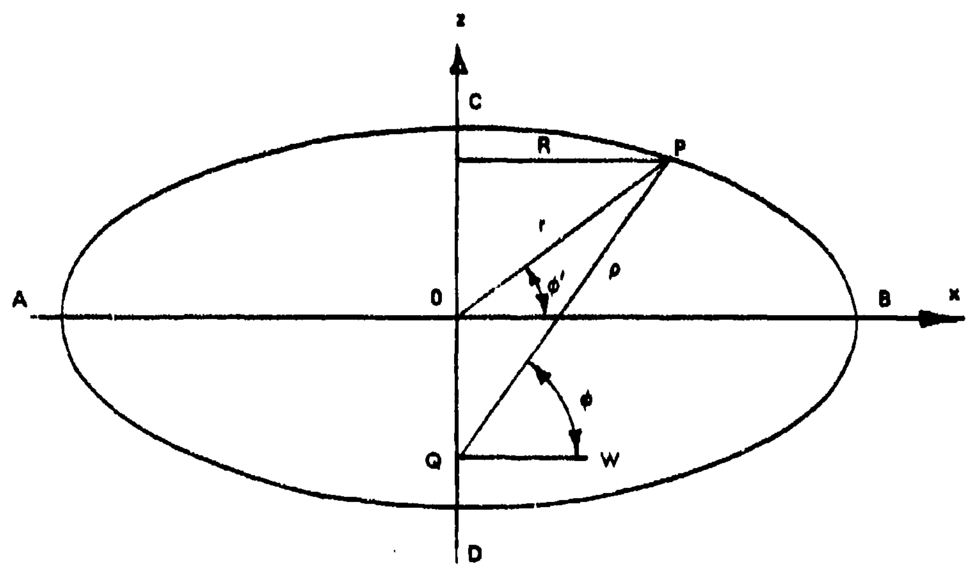


Figure 3.1.1. Geometry of the ellipse

The degree of departure from circularity is described by the eccentricity, e, or the flattening, f. The eccentricity, flattening, semi-major axis, and semi-minor axis are related as follows:

$$e^2 = \frac{a^2 - b^2}{a^2} \tag{3.1.2}$$

$$f = \frac{a - b}{a} \tag{3.1.3}$$

$$e^2 = 2f - f^2 . \tag{3.1.4}$$

At this point we shall introduce one of the two angular coordinates which uniquely locate a position on the spheroid. This first coordinate is the latitude. Two types of latitude will be noted: the geodetic and the geocentric. The relation between the two will be derived.

The geocentric latitude is the angle between a vector from the center of the ellipse, to a point P, on the ellipse, and the semi-major axis. The geodetic latitude is the angle between a line through the given point, normal to the ellipse, and the semi-major axis. This normal to the ellipse is the line defined by a surveyor's plumb line if all gravity anomalies are ignored.

Now, consider a polar coordinate system with the origin at O. The geocentric latitude, ϕ', is $\sphericalangle$COP, and the magnitude of the vector is r. The relation between the Cartesian and the polar coordinates is

$$x = a \cos \phi' \tag{3.1.5}$$

$$z = b \sin \phi' . \tag{3.1.6}$$

Equations (3.1.5) and (3.1.6) can be combined to form

$$\frac{z}{x} = \frac{b}{a} \tan \phi' . \tag{3.1.7}$$

Substitute (3.1.2) into (3.1.7).

$$\frac{z}{x} = \sqrt{1 - e^2} \tan \phi' . \tag{3.1.8}$$

Of greater interest is the geodetic latitude, ϕ. This angle, $\sphericalangle$PQW, defines the inclination of line QP, which is normal to the ellipse at point P.

$$\tan \phi = -\frac{dx}{dz} . \tag{3.1.9}$$

Taking the differential of (3.1.1)

$$\frac{2x\,dx}{a^2} + \frac{2z\,dz}{b^2} = 0$$

$$\frac{dx}{dz} = -\frac{a^2}{b^2}\,\frac{z}{x}. \tag{3.1.10}$$

Substitute (3.1.10) into (3.1.9).

$$\tan\phi = \frac{a^2}{b^2}\,\frac{z}{x}. \tag{3.1.11}$$

Substitute (3.1.7) into (3.1.11).

$$\tan\phi = \frac{a^2}{b^2}\,\frac{b}{a}\,\tan\phi'$$

$$\tan\phi' = \frac{b}{a}\,\tan\phi. \tag{3.1.12}$$

Substitute (3.1.2) into (3.1.12).

$$\tan\phi' = \sqrt{1 - e^2}\,\tan\phi. \tag{3.1.13}$$

Table 3.1.1 gives the relation between geocentric latitude and geodetic latitude for the WGS-72 spheroid, which will be discussed in Section 3.4.

The convention for measuring geodetic latitude is $+\phi$ in the northern hemisphere, and $-\phi$ in the southern hemisphere.

Table 3.1.1. Geocentric and Geodetic Latitude for the WGS-72 Spheroid (Degrees).

Geodetic ϕ	Geocentric ϕ'	Geocentric ϕ'	Geodetic ϕ
0.00	0.0000	0.00	0.0000
5.00	4.9833	5.00	5.0167
10.00	9.9672	10.00	10.0330
15.00	14.9520	15.00	15.0482
20.00	19.9383	20.00	20.0619
25.00	24.9264	25.00	25.0738
30.00	29.9168	30.00	30.0834
35.00	34.9097	35.00	35.0906
40.00	39.9053	40.00	40.0948
45.00	44.9038	45.00	45.0962
50.00	49.9053	50.00	50.0947
55.00	54.9096	55.00	55.0904
60.00	59.9167	60.00	60.0833
65.00	64.9263	65.00	65.0737
70.00	69.9381	70.00	70.0618
75.00	74.9519	75.00	75.0481
80.00	79.9671	80.00	80.0329
85.00	84.9833	85.00	85.0167
90.00	90.0000	90.00	90.0000

3.2 Geometry of the Spheroid [23]

The spheroid, which is taken as the model of the earth, is obtained by revolving the ellipse of Figure 3.1.1 about the z-axis. For the Cartesian coordinate system shown in Figure 3.2.1, the equation of the spheroidal surface is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1 . \tag{3.2.1}$$

The nomenclature of the spheroid can be obtained by studying Figure 3.2.1. Each of the infinity of positions of the ellipse as it is rotated about the z-axis defines a meridianal ellipse, or meridian. The angle λ, measured in the x-y plane, and from the x-axis, is the longitude of any and all points on the meridianal ellipse. This is the second of the two angular coordinates which uniquely define a position on the spheroid. As a convention, a rotation from +x to +y will be positive, and the reverse rotation, negative.

Consider the point P in figure 3.2.1 to be defined by ϕ and λ. Suppose now that λ is allowed to vary, while ϕ is held constant. The locus on the spheroid traced out by P is a circle of parallel of radius R. The circle of parallel for a latitude of zero is the equator.

It remains to derive the equations for several radii of importance in future developments. These are the two principle radii of curvature, and the radius of a parallel circle, all as a function of latitude.

Consider the meridianal ellipse at any arbitrary λ. From (3.2.1)

$$\frac{R_0^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{3.2.2}$$

where $R_0 = \sqrt{x^2 + y^2}$ is the radius of a parallel circle.

Substitute (3.1.2) into (3.2.2).

$$\frac{R_0^2}{a^2} + \frac{z^2}{a^2(1-e^2)} = 1$$

$$R_0^2(1-e^2) + z^2 = a^2(1-e^2) . \tag{3.2.3}$$

Take the differential of (3.2.3) to obtain the slope of the tangent at P.

$$2R_0\, dR_0(1-e^2) + 2z\, dz = 0$$

$$\frac{dz}{dR_0} = -\frac{R_0}{z}(1-e^2) . \tag{3.2.4}$$

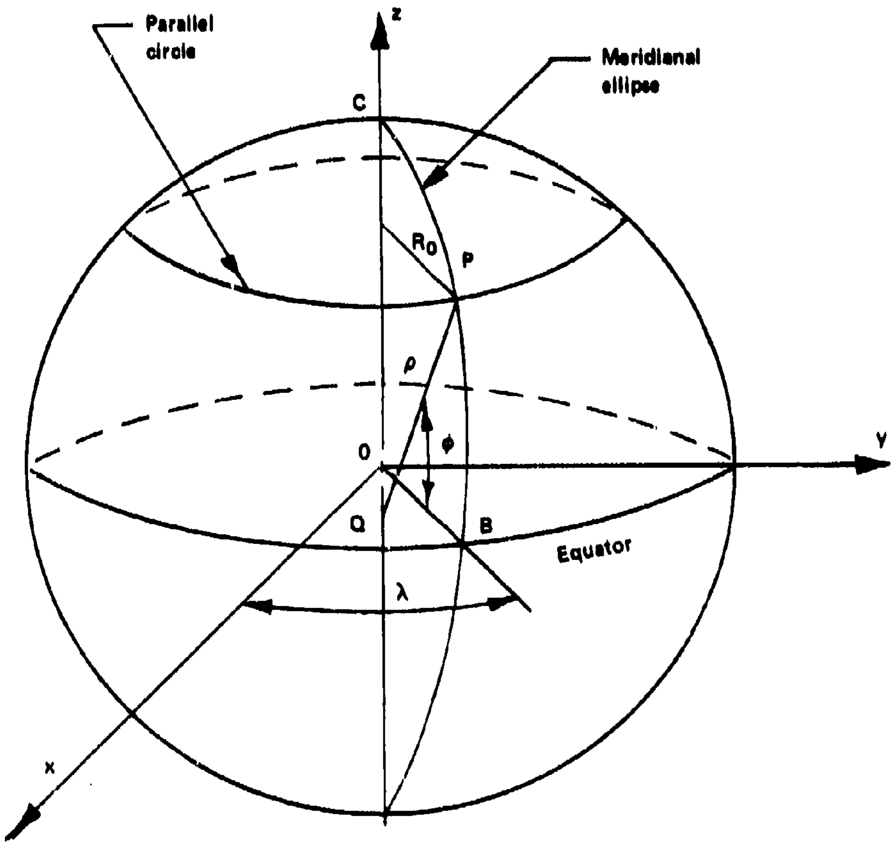


Figure 3.2.1. Geometry of the spheroid

The slope of the normal at P is again

$$-\frac{dR_0}{dz} = \frac{z}{R_0(1-e^2)} = \tan\phi$$

$$z = R_0(1-e^2)\tan\phi\,. \tag{3.2.5}$$

From Figure 3.2.1

$$\sin\phi = \frac{z}{(1-e^2)\overline{QP}}$$

$$z = \overline{QP}(1-e^2)\sin\phi\,. \tag{3.2.6}$$

Substitute (3.2.5) into (3.2.3)

$$R_0^2(1-e^2) + R_0^2(1-e^2)^2\tan^2\phi = a^2(1-e^2)$$

$$R_0^2 + R_0^2(1-e^2)\tan^2\phi = a^2$$

$$(\cos^2\phi + \sin^2\phi)R_0^2 - e^2R_0^2\sin^2\phi = a^2\cos^2\phi$$

$$R_0^2(1-e^2\sin^2\phi) = a^2\cos^2\phi$$

$$R_0 = \frac{a\cos\phi}{\sqrt{1-e^2\sin^2\phi}}\,. \tag{3.2.7}$$

Also, from Figure 3.2.1

$$R_0 = \overline{QR}\cos\phi\,. \tag{3.2.8}$$

Equating (3.2.7) and (3.2.8)

$$\overline{QP}\cos\phi = \frac{a\cos\phi}{\sqrt{1-e^2\sin^2\phi}}$$

$$\overline{QP} = \frac{a}{\sqrt{1-e^2\sin^2\phi}}\,.$$

$\overline{QP}$ is the radius of curvature of the spheroid in the plane perpendicular to the meridianal plane, and will be denoted as R_p.

$$R_p = \frac{a}{\sqrt{1 - e^2 \sin^2 \phi}} . \tag{3.2.9}$$

The radius of curvature of the meridianal ellipse follows from the formula for a plane curve (2.1.35).

$$R_M = \left| \frac{\left[1 + \left(\frac{dz}{dR_0}\right)^2\right]^{3/2}}{\frac{d^2 z}{dR_0^2}} \right| . \tag{3.2.10}$$

From (3.2.4)

$$\frac{d^2 z}{dR^2} = -\frac{1}{z} (1 - e^2) + \frac{R_0}{z^2} (1 - e^2) \frac{dz}{dR} . \tag{3.2.11}$$

Substituting (3.2.4) into (3.2.10)

$$\frac{d^2 z}{dR_0^2} = \frac{1}{z} \left[-(1 - e^2) - \left(\frac{dz}{dR_0}\right)^2 \right]$$

$$= -\frac{1}{z} \left[1 + \left(\frac{dz}{dR_0}\right)^2 - e^2 \right] . \tag{3.2.12}$$

Since the slope of the normal is $\tan\phi$, that of the tangent $-\cot\phi$.

$$\frac{dz}{dR_0} = \cot\phi . \tag{3.2.13}$$

Substituting (3.2.13) into (3.2.12)

$$\frac{dz}{dR_0} = -\frac{1 + \cot^2\phi - e^2}{z}$$

$$= -\frac{\frac{1}{\sin^2\phi} - e^2}{z} . \tag{3.2.14}$$

From (3.2.6) and (3.2.9)

$$z = \frac{a(1-e^2)\sin\phi}{\sqrt{1-e^2\sin^2\phi}}. \tag{3.2.15}$$

Substitute (3.2.13), (3.2.14), and (3.2.15) into (3.2.10).

$$R_m = \left| \frac{\left(1 + \dfrac{\cos^2\phi}{\sin^2\phi}\right)^{3/2}}{-\left(\dfrac{1}{\sin^2\phi} - e^2\right)\dfrac{\sqrt{1-e^2\sin^2\phi}}{a(1-e^2)\sin\phi}} \right|$$

$$= \left| \frac{\left(\dfrac{\cos^2\phi + \sin^2\phi}{\sin^2\phi}\right)^{3/2}}{-\left(\dfrac{1-e^2\sin^2\phi}{\sin^2\phi}\right)\dfrac{\sqrt{1-e^2\sin^2\phi}}{a(1-e^2)\sin\phi}} \right|$$

$$R_m = \left| - \frac{\dfrac{1}{\sin^3\phi}}{\left(\dfrac{1-e^2\sin^2\phi}{\sin^3\phi}\right)^{3/2} \cdot \dfrac{1}{a(1-e^2)}} \right|$$

$$= \left| \frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{3/2}} \right|. \tag{3.2.16}$$

Equations (3.2.7), (3.2.9), and (3.2.16) give the radius of the circle of parallel, the radius of curvature normal to the meridian, and the radius of curvature in the meridianal plane, respectively. The radii of curvature are tabulated in Table 3.2.1 as functions of latitude for the WGS-72 spheroid (Section 3.5).

Before turning to distances on the spheroid, it will be useful to relate the Cartesian coordinates for the spheroid to the polar coordinates.

$$\left.\begin{aligned} x &= R_p \cos\phi \cos\lambda \\ y &= R_p \cos\phi \sin\lambda \\ z &= (1-e^2) R_p \sin\phi \end{aligned}\right\}. \tag{3.2.17}$$

Table 3.2.1. Radii as a Function of Latitude.

Geodetic Latitude ϕ*	Radii of Curvature R_p**	R_m**
0.00	6378165.	6335477.
5.00	6378327.	6335960.
10.00	6378809.	6337395.
15.00	6379595.	6339740.
20.00	6380663.	6342924.
25.00	6381981.	6346854.
30.00	6383508.	6351411.
35.00	6385199.	6356460.
40.00	6387002.	6361847.
45.00	6388864.	6367412.
50.00	6399727.	6372985.
55.00	6392536.	6378396.
60.00	6394234.	6383481.
65.00	6395770.	6388082.
70.00	6397096.	6392058.
75.00	6398173.	6395287.
80.00	6398967.	6397667.
85.00	6399453.	6399126.
90.00	6399617.	6399617.

*Degrees
**Meters

The first fundamental form of the spheroid (Section 2.3) becomes

$$(ds)^2 = R_m^2 (d\phi)^2 + R_p^2 \cos^2\phi (d\lambda)^2 . \tag{3.2.18}$$

Equation (3.2.18) will be useful in the transformations from the spheroid to the sphere in Chapters 4 and 5, and in the discussion of distortions in Chapter 7.

3.3 Distances and Angles on the Spheroid [5]

Three types of distances measured on the spheroid will be considered. These are distances along a circle of parallel, along the meridianal ellipse, and between two ordinary points.

We can deal with the distance along a circle of parallel very easily. From (3.2.7)

$$d = \frac{a\Delta\lambda \cos\phi}{\sqrt{1 - e^2 \sin^2\phi}} \tag{3.3.1}$$

where $\Delta\lambda$ is the angular separation of two points on the circle of parallel, in radians. Table 3.3.1 gives the distance for an angular separation of 1′ as a function of latitude for the WGS-72 ellipsoid.

Distance along the meridianal ellipse requires an integration of (3.2.16). To facilitate this, (3.2.16) is expanded by the binomial theorem.

$$R_m = a(1-e^2)\left(1 + \frac{3}{2} e^2 \sin^2\phi + \frac{15}{8} e^4 \sin^4\phi + \frac{35}{16} e^6 \sin^6\phi + \ldots\right). \tag{3.3.2}$$

Equation (3.3.2) is a rapidly converging series, as we shall see when we display the values of e in Section 3.4.

The distance between positions at latitude ϕ_1 and ϕ_2 on the same meridianal ellipse are

$$d = \int_{\phi_1}^{\phi_2} R_m \, d\phi . \tag{3.3.3}$$

Substitute (3.3.2) into (3.3.3) and integrate

$$d = a(1-e^2) \int_{\phi_1}^{\phi_2} \left(1 + \frac{3}{2} e^2 \sin^2\phi + \frac{15}{8} e^4 \sin^4\phi + \frac{35}{16} e^6 \sin^6\phi + \ldots\right) d\phi$$

Table 3.3.1. Distances Along the Circle of Parallel for a Separation of 1′.

Geodetic Latitude ϕ*	Distance d**
0.00	1855.332
5.00	1848.319
10.00	1827.329
15.00	1792.515
20.00	1744.124
25.00	1682.507
30.00	1608.110
35.00	1521.474
40.00	1423.235
45.00	1314.118
50.00	1194.933
55.00	1066.572
60.00	930.002
65.00	786.260
70.00	636.443
75.00	481.701
80.00	323.225
85.00	162.241
90.00	-.001

*Degrees
**Meters

$$
\begin{aligned}
d = a(1-e^2) & \left\{ \phi + \frac{3}{2} e^2 \left(\frac{\phi}{2} - \frac{1}{4} \sin 2\phi \right) \right. \\
& + \frac{15}{8} e^4 \left(\frac{3\phi}{8} - \frac{\sin 2\phi}{4} + \frac{\sin 4\phi}{32} \right) \\
& \left. + \frac{35}{16} e^6 \left[- \frac{\sin^5 \phi \cos \phi}{6} + \frac{5}{6} \left[\frac{3\phi}{8} - \frac{\sin 2\phi}{4} + \frac{\sin 4\phi}{32} \right] + \ldots \right\} \right|_{\phi_1}^{\phi_2}
\end{aligned}
$$

$$
\begin{aligned}
= a(1-e^2) & \left\{ \phi \left(1 + \frac{3}{4} e^2 + \frac{45}{64} e^4 + \frac{525}{768} e^6 + \ldots \right) \right. \\
& - \sin 2\phi \left(\frac{3}{8} e^2 + \frac{15}{32} e^4 + \frac{175}{384} e^6 + \ldots \right) \\
& + \sin 4\phi \left(\frac{15}{256} e^4 + \frac{175}{3072} e^6 + \ldots \right) \\
& \left. - \frac{35}{96} e^6 \sin^5 \phi \cos \phi + \ldots \right\} \Big|_{\phi_1}^{\phi_2}
\end{aligned}
$$

$$
\begin{aligned}
= a & \left\{ \left(1 - e^2 + \frac{3}{4} e^2 - \frac{3}{4} e^4 + \frac{45}{64} e^4 - \frac{45}{64} e^6 + \frac{525}{768} e^6 + \ldots \right) \phi \right. \\
& - \left(\frac{3}{8} e^2 - \frac{3}{8} e^4 + \frac{15}{32} e^4 - \frac{15}{32} e^6 + \frac{175}{384} e^6 + \ldots \right) \sin 2\phi \\
& + \left(\frac{15}{256} e^4 - \frac{15}{256} e^6 + \frac{175}{3072} e^4 + \ldots \right) \sin 4\phi \\
& \left. - \frac{35}{96} e^6 \left(\frac{\sin \phi \cos \phi}{8} \right) (3 - 4 \cos 2\phi + \cos 4\phi) \right\} \Big|_{\phi_1}^{\phi_2}
\end{aligned}
$$

$$d = a\left\{\left(1 - \frac{e^2}{4} - \frac{3}{64}e^4 - \frac{5}{256}e^6 - \ldots\right)\phi\right.$$

$$-\left(\frac{3}{8}e^2 + \frac{3}{32}e^4 - \frac{5}{384}e^6 + \ldots\right)\sin 2\phi$$

$$+\left(\frac{15}{256}e^4 - \frac{5}{3072}e^6 + \ldots\right)\sin 4\phi$$

$$\left.-\frac{35}{96}e^6\left(\frac{\sin 2\phi}{16}\right)(3 - 4\cos 2\phi + \cos 4\phi)\right\}_{\phi_1}^{\phi_2}. \qquad (3.3.4)$$

Further expand the last term in (3.3.4)

$$-\frac{35}{96}e^6\left(\frac{\sin 2\phi}{16}\right)(3 - 4\cos 2\phi + \cos 4\phi)$$

$$= -\frac{35}{96}e^6\left(\frac{3}{16}\sin 2\phi - \frac{1}{4}\sin 2\phi\cos 2\phi + \frac{1}{16}\sin 2\phi\cos 4\phi\right)$$

$$= -\frac{35}{96}e^6\left(\frac{3}{16}\sin 2\phi - \frac{1}{8}\sin 4\phi - \frac{1}{32}\sin 2\phi + \frac{1}{32}\sin 6\phi\right). \qquad (3.3.5)$$

Substitute (3.3.5) into (3.3.4).

$$d = a\left\{\left(1 - \frac{e^2}{4} - \frac{3}{64}e^4 - \frac{5}{256}e^6 - \ldots\right)\phi\right.$$

$$-\left[\frac{3}{8}e^2 + \frac{3}{32}e^4 + \left(-\frac{5}{384} + \frac{35}{96}\left(\frac{3}{16} - \frac{1}{32}\right)\right)e^6 + \ldots\right]\sin 2\phi$$

$$+\left[\frac{15}{256}e^4 + \left(-\frac{5}{3072} + \frac{35}{96}\cdot\frac{1}{8}\right)e^6 + \ldots\right]\sin 4\phi$$

$$\left.-\frac{35}{96}\cdot\frac{e^6}{32}\sin 6\phi + \ldots\right\}_{\phi_1}^{\phi_2}$$

$$d = a\left\{\left(1 - \frac{e^2}{4} - \frac{3}{64}e^4 - \frac{5}{256}e^6 - \ldots\right)\phi\right.$$

$$-\left(\frac{3}{8}e^2 + \frac{3}{32}e^4 + \frac{45}{1024}e^6 + \ldots\right)\sin 2\phi$$

$$\left.+\left(\frac{15}{256}e^4 + \frac{45}{1024}e^6 + \ldots\right)\sin 4\phi - \frac{35}{3072}e^6 \sin 6\phi + \ldots\right\}_{\phi_1}^{\phi_2} \qquad (3.3.6)$$

Equation (3.3.6) has been evaluated in Table 3.3.2 for 1′ intervals of arc along the meridional ellipse as a function of latitude for the WGS-72 spheroid.

The distance along a spheroid between two arbitrary points $P_1(\phi_1, \lambda_1)$ and $P_2(\phi_2, \lambda_2)$ is obtained from consideration of the first fundamental form for a spheroid (3.2.18).

$$s = \int_{\phi_1}^{\phi_2} \left[R_m^2 + R_p^2 \cos^2\phi \left(\frac{d\lambda}{d\phi}\right)^2\right]^{1/2} d\phi\,. \qquad (3.3.7)$$

To obtain the shortest line on the spheroid connecting P_1 and P_2, that is, the geodesic curve, apply the Euler-Lagrange relations to (3.3.7), where

$$L(\phi, \lambda, \lambda') = R_m^2 + R_p^2 \cos^2\phi(\lambda')^2$$

$$\frac{d}{d\phi}\left(\frac{\partial L}{\partial \lambda'}\right) = \frac{\partial L}{\partial \lambda} = 0$$

$$\frac{\partial L}{\partial \lambda'} = R_p^2 \cos^2\phi \frac{d\lambda}{d\phi} = c \qquad (3.3.8)$$

where c is a constant.

Substitute (3.3.8) into (3.3.7).

$$s = \int_{\phi_1}^{\phi_2} \left(R_m^2 + \frac{c^2}{R_p^2 \cos^2\phi}\right)^{1/2} d\phi\,. \qquad (3.3.9)$$

It remains to evaluate c in (3.3.9). Integrate (3.3.8).

$$\lambda = c \int \frac{d\phi}{R_p^2 \cos^2\phi} + k\,. \qquad (3.3.10)$$

Substitute (3.2.9) into (3.3.10).

Table 3.3.2. Distance Along the Meridianal Ellipse for a Separation of 1'.

Geodetic Latitude ϕ*	Distance d**
0.00	1842.914
5.00	1843.055
10.00	1843.472
15.00	1844.154
20.00	1845.081
25.00	1846.224
30.00	1847.549
35.00	1849.018
40.00	1850.585
45.00	1852.204
50.00	1853.825
55.00	1855.399
60.00	1856.878
65.00	1858.216
70.00	1859.373
75.00	1860.312
80.00	1861.005
85.00	1861.429
90.00	1861.572

*Degrees
**Meters

$$\lambda = c \int \left(\frac{1 - e^2 \sin^2 \phi}{a^2 \cos^2 \phi} \right) d\phi + k$$

$$= c \int \left[\frac{1 - e^2 (1 - \cos^2 \phi)}{a^2 \cos^2 \phi} \right] d\phi + k$$

$$= c \int \left(\frac{1 - e^2}{a^2 \cos^2 \phi} + \frac{e^2}{a^2} \right) d\phi + k$$

$$= c \left[\left(\frac{1 - e^2}{a^2} \right) \tan \phi + \frac{e^2 \phi}{a^2} \right] + k \,. \tag{3.3.11}$$

Evaluate (3.3.11) at P_1 and P_2, and subtract to eliminate k.

$$\lambda_1 = c \left[\left(\frac{1 - e^2}{a^2} \right) \tan \phi_1 + \frac{e^2}{a^2} \phi_1 \right] + k$$

$$\lambda_2 = c \left[\left(\frac{1 - e^2}{a^2} \right) \tan \phi_2 + \frac{e^2}{a^2} \phi_2 \right] + k$$

$$\lambda_2 - \lambda_1 = c \left[\left(\frac{1 - e^2}{a^2} \right) (\tan \phi_2 - \tan \phi_1) + \frac{e^2}{a^2} (\phi_2 - \phi_1) \right]$$

$$c = \frac{\lambda_2 - \lambda_1}{\left(\frac{1 - e^2}{a^2} \right) (\tan \phi_2 - \tan \phi_1) + \frac{e^2}{a^2} (\phi_2 - \phi_1)} \,. \tag{3.3.12}$$

Substitute (3.3.9), (3.2.16) and (3.3.12) into (3.3.9).

$$s = \int_{\phi_1}^{\phi_2} \left\{ \frac{a^2 (1 - e^2)}{(1 - e^2 \sin^2 \phi)^3} + \left(\frac{1 - e^2 \sin^2 \phi}{a^2 \cos^2 \phi} \right) \right.$$

$$\left. \times \frac{(\lambda_2 - \lambda_1)^2}{\left[\left(\frac{1 - e^2}{a^2} \right) (\tan \phi_2 - \tan \phi_1) - \frac{e^2}{a^2} (\phi_2 - \phi_1) \right]^2} \right\}^{1/2} d\phi$$

$$s = a \int_{\phi_1}^{\phi_2} \left\{ \frac{(1-e^2)^2}{(1-e^2 \sin^2\phi)^3} + \frac{(\lambda_2-\lambda_1)^2(1-e^2\sin^2\phi)}{[(1-e^2)(\tan\phi_2-\tan\phi_1) - e^2(\phi_2-\phi_1)]^2\cos^2\phi} \right\}^{1/2} d\phi \,. \qquad (3.3.13)$$

Faced with a horrid equation such as (3.3.13), the only reasonable procedure is a numerical integration on a computer for a specific choice of starting and ending points. The equation itself is completely general.

Several features of the geodesic will be noted. In general, the geodesic is not a plane curve. However, the plane which contains any three near points on it also contains the normal to the spheroid at the center point of the three. The meridianal ellipse is one particular geodesic, and is obtained by setting $d\lambda/d\phi = 0$ in (3.3.7). The equator is also a particular geodesic. The meridians and the equator are the only geodesics which are plane curves.

Another feature, is that along any geodesic, $R_p \cos\phi \sin\alpha$ is constant, where α is the azimuth.

Three differential formulas will be derived which apply at any point on the geodesic, and relate ϕ, λ, α, and s, where α is the azimuth.

$$\frac{d\phi}{ds} = \frac{\cos\alpha}{R_m} \qquad (3.3.14)$$

$$\frac{d\lambda}{ds} = \frac{1}{R_p}\,\frac{\sin\alpha}{\cos\phi} \qquad (3.3.15)$$

$$\frac{d\alpha}{ds} = \frac{1}{R_p}\,\tan\phi\sin\alpha \,. \qquad (3.3.16)$$

Equations (3.3.14) and (3.3.15) can be obtained from a consideration of the angle between a curve on the spheroidal surface, and one of the parametric curves, the meridianal ellipse, where λ is a constant.

From the first fundamental form for the spheroid,

$$E = R_m^2 \qquad (3.3.17)$$

$$G = R_p^2 \cos^2\phi \qquad (3.3.18)$$

$$F = 0 \,. \qquad (3.3.19)$$

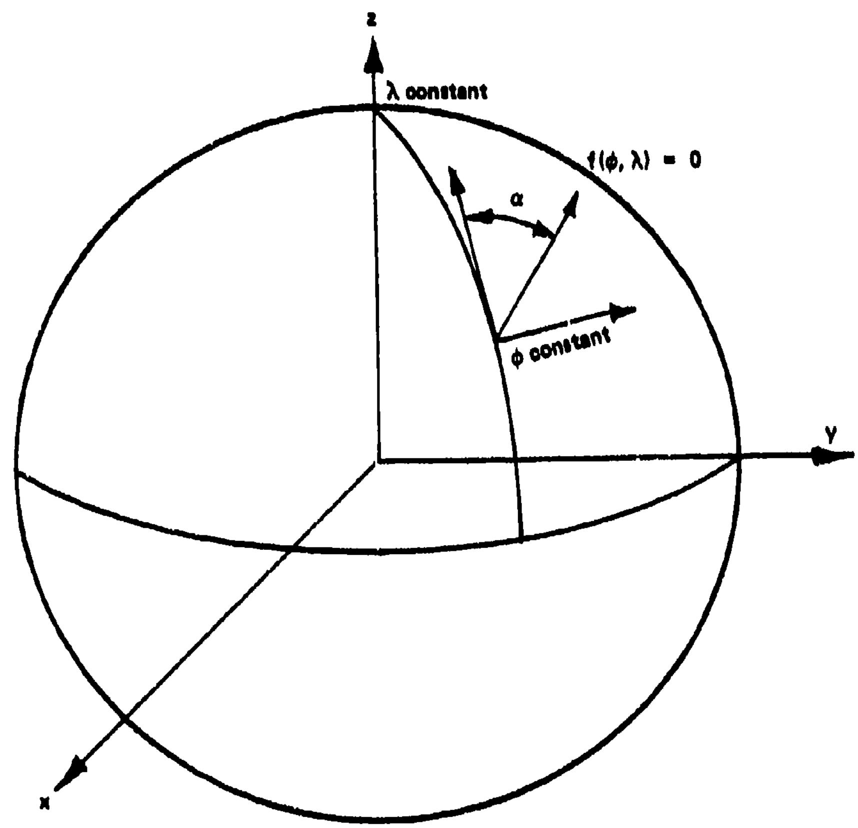


Figure 3.3.1. The azimuth of a curve

Since $\lambda_1 = c$

$$d\lambda_1 = 0 \tag{3.3.20}$$

and

$$ds_1 = \sqrt{E}\, d\phi_1 \tag{3.3.21}$$

$$\cos\alpha = \frac{E\, d\phi\, d\phi_1 + G\, d\lambda\, d\lambda_1}{ds\, ds_1}\,. \tag{3.3.22}$$

Substitute (3.3.20) and (3.3.21) into (3.3.22).

$$\cos\alpha = \frac{E}{\sqrt{E}} \frac{d\phi}{ds} \frac{d\phi_1}{d\phi_1}$$

$$= \sqrt{E}\, \frac{d\phi}{ds}\,. \tag{3.3.23}$$

Substitute (3.3.17) into (3.3.23).

$$\cos\alpha = R_m \frac{d\phi}{ds}\,.$$

This has justified (3.3.14).

$$\sin\alpha = \sqrt{EG} \left(\frac{d\phi_1\, d\lambda - d\phi\, d\lambda_1}{ds\, ds_1}\right)\,. \tag{3.3.24}$$

Substitute (3.3.20) and (3.3.21) into (3.3.24).

$$\sin\alpha = \frac{\sqrt{EG}}{\sqrt{E}} \frac{d\phi_1\, d\lambda}{d\phi_1\, ds}$$

$$= \sqrt{G}\, \frac{d\lambda}{ds}\,. \tag{3.3.25}$$

Substitute (3.3.18) into (3.3.25).

$$\sin\alpha = R_p \cos\phi\, \frac{d\lambda}{ds}\,.$$

This is (3.3.15).

From (3.3.14) and (3.3.13), the azimuth at P at the initiation of the geodesic can be calculated.

$$\cos\alpha_1 = R_m \frac{d\phi}{ds}$$

$$= (R_m)_1 \left\{ \frac{(1-e^2)^2}{(1-e^2\sin^2\phi_1)^3} \right.$$

$$\left. + \frac{(\lambda_2-\lambda_1)^2(1-e^2\sin^2\phi_1)}{[(1-e^2)(\tan\phi_2-\tan\phi_1) - e^2(\phi_2-\phi_1)]^2\cos^2\phi_1} \right\}^{1/2} . \qquad (3.3.26)$$

The rhumbline (or loxodrome) is a curve on the spheroid which meets each consecutive meridian at the same azimuth. From Figure 3.3.2,

$$\tan\alpha = \frac{R_p}{R_m\, d\phi} \cos\phi\, d\lambda$$

$$d\lambda = \tan\alpha \frac{R_p}{R_m \cos\phi} d\phi . \qquad (3.3.27)$$

Substitute (3.2.9) and (3.2.16) into (3.3.17),

$$d\lambda = \tan\alpha \frac{\dfrac{a}{\sqrt{1-e^2\sin^2\phi}}}{\dfrac{a(1-e^2)\cos\phi}{(1-e^2\sin^2\phi)^{3/2}}} d\phi$$

$$= \tan\alpha \left[\frac{1-e^2\sin^2\phi}{(1-e^2)\cos\phi}\right] d\phi$$

$$\Delta\lambda = \tan\alpha \left[\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1-e\sin\phi}{1+e\sin\phi}\right)^{e/2}\right] . \qquad (3.3.28)$$

The kernal of (3.3.18) will be seen again when we treat the conformal projections of Chapter 5. The rhumbline, used in conjunction with the Mercator projection (Chapter 5) and the great circle on the gnomonic projection (Chapter 6) are longstanding aids to marine and aerial navigation.

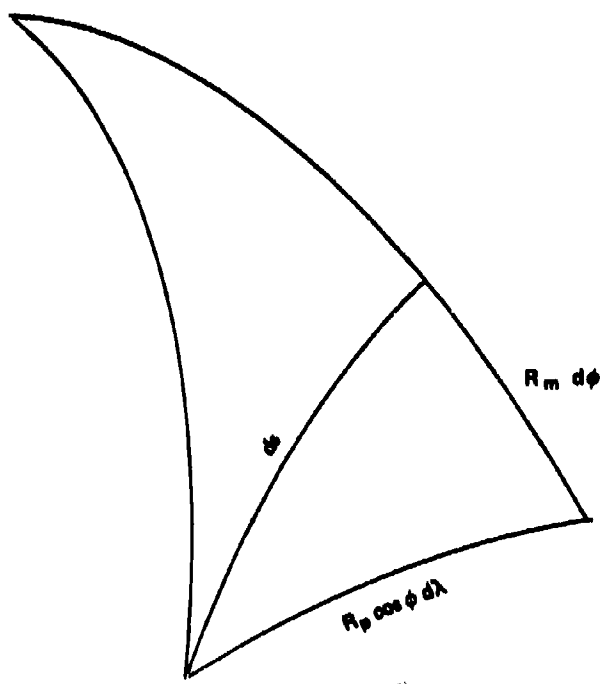


Figure 3.3.2. Differential element defining a rhumbline on a spheroid

3.4 Geodetic Spheroids [5], [7], [18], [21]

Beginning in the early 1800's, a serious attempt was made to find the correct dimensions of the earth. Geodesists and surveyors undertook to find the best representation.

The physical shape of the earth is too irregular to be used in any mathematical study. Thus, it was necessary to define a fictitious surface which approximates the total shape of the earth. The surfaces of revolution of this section are the geodesists answer to the problem. They are the most convenient surfaces which best fit the true figure of the earth.

The geodesist's first approximation to the shape of the earth is the equipotential surface at mean sea level called the geoid. The geoid is smooth and continuous, and extends under the continents at the continued mean sea level. By definition, the perpendicular at any point of the geoid is in the direction of the gravity vector. This surface, however, is not symmetrical about the axis of revolution, since the distribution of matter within the earth is not uniform. The geoid is the intermediate projection surface between the irregular earth, and the mathematically manageable surface of revolution.

The estimates of the values of the semi-major axis and the flattening have changed over the last 145 years. Progress has meant better instruments, better methods of their use, and better methods of choosing the best fit between the geoid and the spheroid. The instruments, their use, and the statistical methods of reducing and fitting the data are beyond the scope of this report. Nevertheless, we must be aware of their existence, and their contribution to mapping.

Table 3.4.1 gives the names and dates of important reference spheroids, as well as the equatorial semi-major axis, a, and the flattening, f, of the ellipse. Historically, the first of these spheroids, the Everest through the Clarke 1880, were intended to fit local areas of the world. Beginning with the Hayford spheroid, an attempt was made to obtain an internationally acceptable representation of the entire world.

Progress has continued in refining the values of the semi-major axis and the flattening. The best representations available today are the World Gravity System of 1972 (WGS-72) and the International Union of Geodesy and Geophysics of 1975 (I.U.G.G.) values. The WGS-72 values will be used in this report.

Unfortunately interest in generating tables such as Table 3.2.1, 3.3.1, 3.3.2, and 4.1.1, and the plotting tables of the projections has flagged since those similar tables incorporating the Clarke 1880 and Hayford spheroids were published [25]. Thus, the tables included in this report, using the WGS-72 spheroid, are the most recent representations of cartographic data available.

Consider now the WGS-72 spheroid. Using (3.1.4), the eccentricity of the meridional ellipse is 0.081819. From (3.1.2)

$$b^2 = a^2(1 - e^2) . \qquad (3.4.1)$$

Using WGS-72 parameters, b = 6356750 meters. Thus, the difference in length between the equatorial and polar axes is 21385 meters.

Table 3.4.1. Reference Spheroids.

Reference Spheroid	Date	a (meters)	f
EVEREST	1830	6377304	1/300.8
BESSEL	1841	6377397	1/299.2
AIRY	1858	6377563	1/299.33
CLARKE	1858	6378294	1/294.3
CLARKE	1866	6378206	1/295
CLARKE	1880	6378249	1/293.5
HAYFORD	1910	6378388	1/292.0
KRASOVSKY	1938	6378245	1/298.3
HOUGH	1956	6378270	1/297.0
FISCHER	1960	6378166	1/298.3
KAULA	1961	6378165	1/292.3
I.U.G.G.	1967	6378160	1/298.25
FISCHER	1968	6378150	1/292.3
WGS-72	1972	6378135	1/298.26
I.U.G.G.	1975	6378140	1/298.257

The eccentricity figures in the series expansions developed in this chapter. It has been noted that these expansions are rapidily convergent, due to the small value of e. This is now demonstrated by Table 3.4.2. In the table are powers of e for the WGS-72 spheroid. Note that considering the seven significant figures for a, it is not necessary to carry any expansion beyond e^6.

Table 3.4.2. Powers of e for the WGS-72 Spheroid.

n	e^n	n	e^n
1	0.081819	4	0.00004481
2	0.00669435	5	0.000003667
3	0.00054772	6	0.000000300

3.5 Reduction of the Formulas to the Sphere [22]

The most simplified model of the earth is the sphere. In the spherical case, the generating curve for the surface is the circle, which is an ellipse of zero eccentricity. The semi-major axis and semi-minor axis are the same. This offers immediate simplifications for the problems of distance, and angular measure. The spheroidal formulas, in gen… l, can be reduced to the spherical by substitution of $e = 0$. However, the formulas for distance between arbitrary points, and azimuth can best be approached by spherical trigonometry.

The equation for the sphere in Cartesian coordinates is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{a^2} = 1 . \qquad (3.5.1)$$

Figure 3.5.1 gives the geometry of the spherical earth. Note that the normal to the sphere at a point, P, coincides with the geocentric radius vector.

For the sphere, there is only one type of latitude, since geocentric and geodetic latitudes coincide. Longitude is measured in the same way it was for the spheroidal case. The sign conventions for latitude and longitude in the spheroidal case also holds for the spherical case.

The radius of a circle of parallel becomes, by substituting $e = 0$ into (3.2.7)

$$R = a \cos\phi . \qquad (3.5.2)$$

By the same substitution, the radii of curvature (3.2.9) and (3.2.16) are found to be

$$R_p = R_m = a . \qquad (3.5.3)$$

The relation between Cartesian and polar coordinates follow from (3.2.17).

$$\left.\begin{aligned} x &= a \cos\phi \cos\lambda \\ y &= a \cos\phi \sin\lambda \\ z &= a \sin\phi \end{aligned}\right\} . \qquad (3.5.4)$$

The first fundamental form is, from (3.2.18)

$$(ds)^2 = a^2 (d\phi)^2 + a^2 \cos^2\phi (d\lambda)^2 . \qquad (3.5.5)$$

Distance along the circle of parallel is, from (3.3.1)

$$d = a\Delta\lambda \cos\phi . \qquad (3.5.6)$$

From (3.3.6), distance along the meridian circle is simply

$$d = a\Delta\phi . \qquad (3.5.7)$$

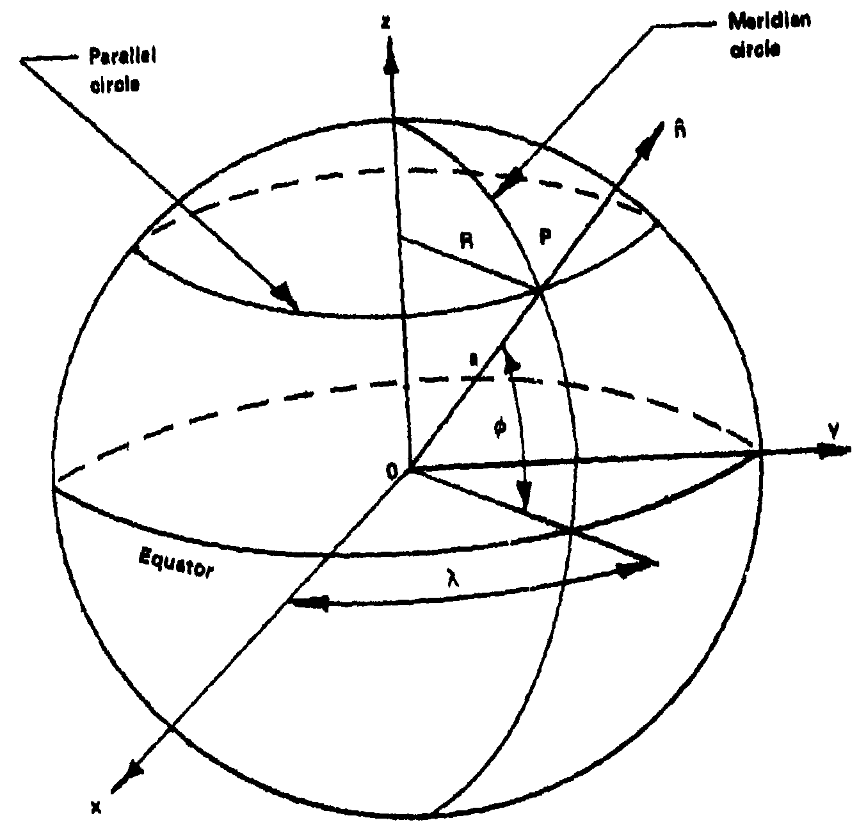


Figure 3.5.1. Geometry of the sphere

On the sphere, the shortest curve connecting two arbitrary points is an arc of the great circle. The great circle (also called the orthodrome) corresponds to the geodesic curve on the spheroid, but with many simplifications. The great circle is a planar curve which contains the two arbitrary points, and the center of the sphere. The distance on the surface of the sphere, can be obtained from consideration of Figure 3.5.2.

The equations of spherical trigonometry will be used to derive the distance, d. Consider the law of cosines.

$$\begin{aligned}\cos\theta &= \cos(90° - \phi_1)\cos(90° - \phi_2) \\ &\quad + \sin(90° - \phi_1)\sin(90° - \phi_2)\cos\Delta\lambda \\ &= \sin\phi_1 \sin\phi_2 \\ &\quad + \cos\phi_1 \cos\phi_2 \cos\Delta\lambda \,. \end{aligned} \tag{3.5.8}$$

Taking the arc-cosine of (3.5.8)

$$d = a\cos^{-1}[\sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos\Delta\lambda] \,. \tag{3.5.9}$$

The azimuth, α, of point P_1 from P_2 is also obtained from the spherical triangle $NP_1 P_2$. Taking the arc-cosine of (3.5.9) again, the angle θ is now available. Then, the law of sines is applied.

$$\frac{\sin\alpha}{\sin(90° - \phi_2)} = \frac{\sin\Delta\lambda}{\sin\theta}$$

$$\sin\alpha = \frac{\cos\phi_2 \sin\Delta\lambda}{\sin\theta} \,. \tag{3.5.10}$$

Also,

$$\cos\alpha = \cos\theta\cos(90° - \phi_1) + \sin\theta\sin(90° - \phi_1)\cos(90° - \phi_2)$$

$$\cos\alpha = \cos\theta\sin\phi_1 + \sin\theta\cos\phi_1\sin\phi_2 \,. \tag{3.5.11}$$

From (3.5.10) and (3.5.11), the quadrant of the azimuth can be seen. As was mentioned in Chapter 1, azimuth is measured from the North, positive to the East, and negative to the West.

The rhumbline or loxodrome is obtained from (3.3.28) by substituting in (3.5.3)

$$\Delta\lambda = \tan\alpha\left[\ln\tan\left(\frac{\pi}{4} + \frac{\phi_2}{2}\right) - \ln\tan\left(\frac{\pi}{4} + \frac{\phi_1}{2}\right)\right] . \tag{3.5.12}$$

Equation (3.5.12) can be investigated. If $\phi_1 = \phi_2$, then $\tan\alpha = 0$, $\alpha = 90$. This is an azimuth along a parallel circle. If $\lambda_1 = \lambda_2$, $\tan\alpha = 0$, $\alpha = 0$, yielding a meridian.

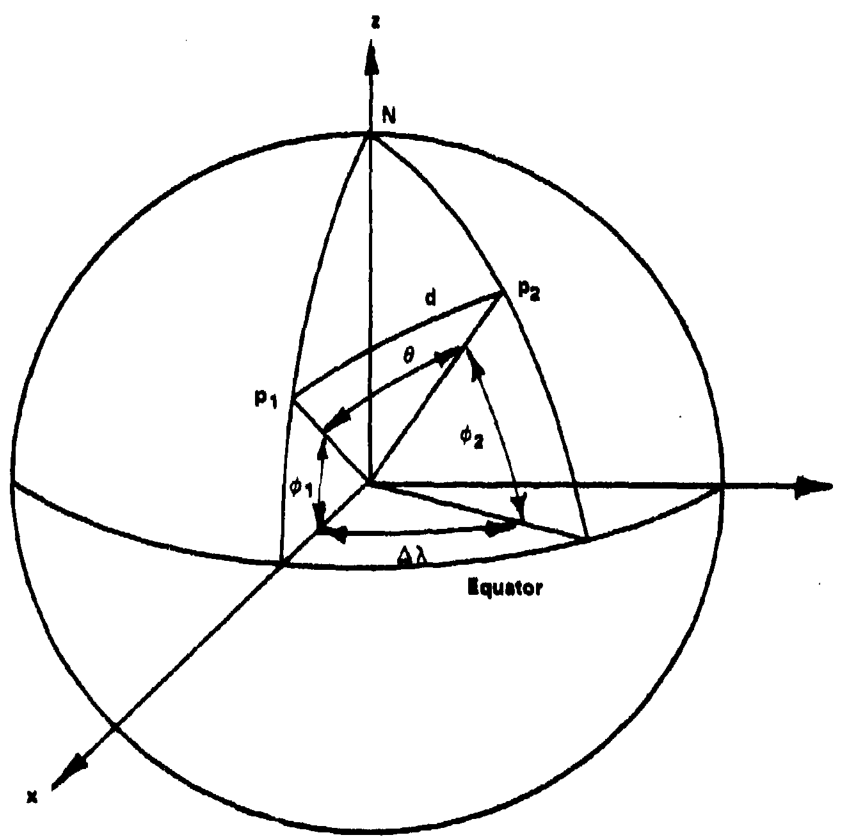


Figure 3.5.2. Distance between arbitrary points on the sphere

The distance along the rhumbline is found from Figure 3.5.3.

$$s = \int_{P_1}^{P_2} ds = \int_{P_1}^{P_2} \frac{R}{\cos\alpha}\, d\phi$$

$$= \frac{R}{\cos\alpha}(\phi_2 - \phi_1)\,. \tag{3.5.13}$$

As was mentioned before, the great circle and loxodrome will be mentioned again in conjunction with the Mercator and gnomonic projections.

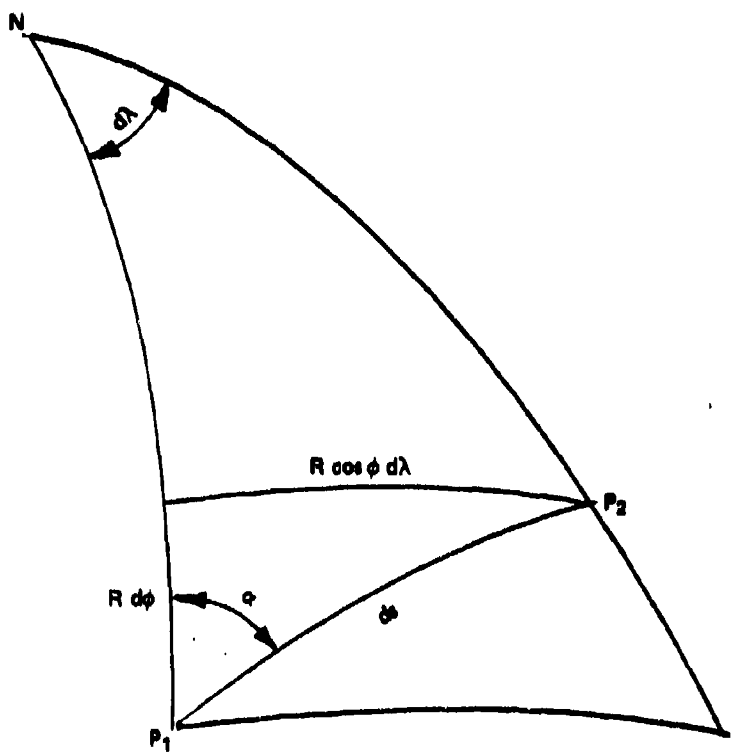


Figure 3.5.3. Differential element defining a rhumbline on a sphere

3.6 Figure of the Moon [4]

With the Space Age, the moon has become a practical entity. Beginning with the U.S. Air Force Mosaic in 1960, engineers and scientists have mapped the moon, the first attempt being an Orthographic projection (Chapter 6).

The moon itself is clearly a triaxial ellipsoid, not a spheroid of revolution. Table 3.6.1(a) gives the semi-axes of the triaxial moon.

However, for mapping convention, the moon is considered to be a spheroid of revolution. The constants of the generating ellipse are given in the Table 3.6.1(b).

Once this simplification is made, all the formulas for the terrestrial spheroid apply immediately. Then, the methods of projection to be developed in Chapters 4, 5, and 6 can be used.

These same techniques also apply to mapping other planets. No doubt, the Viking pictures of July 1976 will be analyzed and reduced to provide a new map of Mars using one of the projections now to be considered.

Table 3.6.1. Lunar Spheroid

(a) Actual

a*	b*	c*
1738570	1738210	1737490

(b) Conventional

a*	f
1738390	1/1031.5

*Meters

Chapter 4

EQUAL AREA PROJECTIONS

The requirement which underlies all the projections in this chapter is stated as follows: Every section of the resulting map must bear a constant ratio to the area of the earth represented by it. This requirement will be stated in mathematical terms. Thus, all of the projections of this chapter are founded on some algorithm which maintains the equivalency of area.

As in any endeavor, there is a hard way and an easy way to work. The cartographers of days past attempted to transform from the spheroid directly to the developable surface. The easier, and more modern approach, is to transform from the spheroid to the equivalent area sphere, and then transform from the sphere to the developable surface. This results in equations which are far less cumbersome. This chapter follows the modern approach.

First, a transformation is derived which defines an authalic sphere. This authalic sphere has the same total area as the spheroid. The longitude of points is undisturbed by the transformation. However, the transformation requires the definition of an authalic latitude on the sphere, which corresponds to the geodetic latitude on the spheroid. Also, the radius of the authalic sphere must be determined.

Second, positions transformed to the authalic sphere are then transformed onto selected developable surfaces to form a map.

The projections to be discussed are the Azimuthal, Conical, and Cylindrical Equal Area, the Bonne, the Werner, and a selection of world maps: the Sinusoidal, Mollweide, Parabolic, Eumorphic, Eckart, and Hammer-Aitoff. In addition, a simple means to minimize extreme distortion is advanced in the Interrupted projections.

A quantitative over-view of the theory of distortions is put off until Chapter 7, where this theory will be applied to the most useful of the projections en masse. Plotting tables for selected projections are given. The computer program which generated the tables is Appendix A.1.

4.1 Authalic Latitude [2], [8], [20]

Authalic latitude is defined by the equal area projection of the spheroid onto a sphere.

From the fundamental transformation matrix of Chapter 2, and the condition of equivalency of area:

$$eg - f^2 = \begin{vmatrix} E & F \\ F & G \end{vmatrix} \begin{vmatrix} \dfrac{\partial \phi_A}{\partial \phi} & \dfrac{\partial \phi_A}{\partial \lambda} \\ \dfrac{\partial \lambda_A}{\partial \phi} & \dfrac{\partial \lambda_A}{\partial \lambda} \end{vmatrix}^2 . \tag{4.1.1}$$

In (4.1.1), ϕ_A and λ_A are the authalic latitude and longitude, respectively, on the authalic sphere, and ϕ and λ are the geodetic latitude and longitude, respectively, on the spheroid. R_A is the radius of the authalic sphere.

Both the systems are orthogonal, so

$$f = F = 0 . \tag{4.1.2}$$

On the spheroid from (2.3.15):

$$ds^2 = R_m^2 \, d\phi^2 + R_p^2 \cos^2\phi \, d\lambda^2$$

$$\left. \begin{aligned} e &= R_m^2 \\ g &= R_p^2 \cos^2\phi \end{aligned} \right\} . \tag{4.1.3}$$

On the authalic sphere from (2.3.14):

$$ds^2 = R_A^2 \, d\phi_A^2 + R_A^2 \cos^2\phi_A \, d\lambda_A^2$$

$$\left. \begin{aligned} E &= R_A^2 \\ G &= R_A^2 \cos^2\phi_A \end{aligned} \right\} . \tag{4.1.4}$$

Substitute (4.1.2), (4.1.3), and (4.1.4) into (4.1.1).

$$R_m^2 \, R_p^2 \cos^2\phi = R_A^4 \cos^2\phi_A \begin{vmatrix} \dfrac{\partial \phi_A}{\partial \phi} & \dfrac{\partial \phi_A}{\partial \lambda} \\ \dfrac{\partial \lambda_A}{\partial \phi} & \dfrac{\partial \lambda_A}{\partial \lambda} \end{vmatrix}^2 . \tag{4.1.5}$$

The longitude is invariant under the transformation: $\lambda = \lambda_A$. Thus,

$$\frac{\partial \lambda_A}{\partial \lambda} = 1 \tag{4.1.6}$$

$$\frac{\partial \lambda_A}{\partial \phi} = 0 . \tag{4.1.7}$$

The authalic latitude is independent of λ_A.

$$\frac{\partial \phi_A}{\partial \lambda} = 0 . \tag{4.1.8}$$

Substitute (4.1.6), (4.1.7), and (4.1.8) into (4.1.5)

$$R_m^2 R_p^2 \cos^2\phi = R_A^4 \cos^2\phi_A \begin{vmatrix} \dfrac{\partial \phi_A}{\partial \phi} & 0 \\ 0 & 1 \end{vmatrix}^2$$

$$R_m^2 R_p^2 \cos^2\phi = R_A^2 \cos\phi_A \left(\frac{\partial \phi_A}{\partial \phi}\right) . \tag{4.1.9}$$

Equation (4.1.9) can now be converted into an ordinary differential equation.

$$R_m R_p \cos\phi \, d\phi = R_A^2 \cos\phi_A \, d\phi_A . \tag{4.1.10}$$

Apply the values of R_m and R_p derived in Chapter 2.

$$R_m = \frac{a(1-e^2)}{(1-e^2 \sin^2\phi)^{3/2}} \tag{2.2.16}$$

$$R_p = \frac{a}{(1-e^2 \sin^2\phi)^{1/2}} . \tag{2.2.17}$$

Substitute (2.2.16) and (2.2.17) into (4.1.10).

$$\frac{a^2(1-e^2)}{(1-e^2 \sin^2\phi)^2} \cos\phi \, d\phi = R_A^2 \cos\phi_A \, d\phi_A . \tag{4.1.11}$$

Integrate (4.1.11)

$$R_A^2 = \int_0^{\phi_A} \cos\phi_A \, d\phi_A = R_A^2 \sin\phi_A$$

$$= a^2(1-e^2) \int_0^{\phi} \frac{\cos\phi}{(1-e^2\sin^2\phi)^2} \, d\phi \, . \qquad (4.1.12)$$

The easiest way to attack (4.1.12) is by means of a binomial expansion:

$$R_A^2 \sin\phi_A = a^2(1-e^2) \int_0^{\phi} \cos\,(1 + 2e^2\sin^2\phi + 3e^4\sin^4\phi + 4e^6\sin^6\phi + \ldots)\, d\phi$$

$$= a^2(1-e^2) \;\; \sin\phi + \frac{2}{3} e^2 \sin^3\phi + \frac{3}{5} e^4 \sin^5\phi + \frac{4}{7} e^6 \sin^7\phi + \ldots \, . \qquad (4.1.13)$$

In order to determine R_A, we introduce the condition that $\phi_A = \pi/2$ when $\phi = \pi/2$. Then, (4.1.13) becomes

$$R_A^2 = a^2(1-e^2)\left(1 + \frac{2}{3} e^2 + \frac{3}{5} e^4 + \frac{4}{7} e^6 + \ldots \right) \qquad (4.1.14)$$

Equation (4.1.14) gives the radius of the authalic sphere with an area equivalent to that of the spheroid.

Substitute (4.1.14) into (4.1.13) to obtain the relation between authalic latitude and geodetic latitude.

$$\sin\phi_A = \sin\phi \left[\frac{1 + \frac{2}{3} e^2 \sin^2\phi + \frac{3}{5} e^4 \sin^4\phi + \frac{4}{7} e^6 \sin^6\phi + \ldots}{1 + \frac{2}{3} e^2 + \frac{3}{5} e^4 + \frac{4}{7} e^6 + \ldots}\right] . \qquad (4.1.15)$$

As was seen in Chapter 3, the eccentricity, e, is a small number for all of the accepted spheroids. Thus, (4.1.15) contains rapidly converging series. The relation between authalic and geodetic latitudes is tabulated in Table 4.1.1 for the WGS-72 Reference Ellipsoid, in increments of 5°.

Now that the transformation from the spheroid to the sphere is completed, the transformations from the sphere to the developable surfaces will be derived. In these derivations, the subscript A on the latitude, longitude, and radius of the authalic sphere will be dropped, and ϕ, λ, and R will be the latitude, longitude, and radius, respectively, of the authalic sphere.

Table 4.1.1. Authalic and Geodetic Latitudes.

Geodetic Latitude (Degrees)	Authalic Latitude (Degrees)
0.00000	0.00000
5.00000	4.97770
10.00000	9.95608
15.00000	14.93577
20.00000	19.91741
25.00000	24.90153
30.00000	29.88863
35.00000	34.87909
40.00000	39.87320
45.00000	44.87114
50.00000	49.87298
55.00000	54.87867
60.00000	59.88802
65.00000	64.90077
70.00000	69.91651
75.00000	74.93477
80.00000	79.95500
85.00000	84.97657
90.00000	89.99874

Radius of the authalic sphere = 6,371,037 meters

4.2 Conical Projections [2], [20], [22]

Two conical equal area projections will be considered. In the first, a cone is tangent to the authalic sphere at a single parallel of latitude. In the second, a cone is secant to a sphere, cutting it at two parallels of latitude. In both cases, the axis of the cone coincides with an extension of the polar axis of the sphere. As will be seen in the following paragraphs, the parallels at the points of tangency or secancy will be the only true length lines on the map.

Two methods of approach will be given for each of the projections.

The first is the differential geometry approach. The line element on the authalic sphere is:

$$ds^2 = R^2\, d\phi^2 + R^2 \cos^2\phi\, d\lambda^2$$

$$\left.\begin{aligned} e &= R^2 \\ g &= R^2 \cos^2\phi \end{aligned}\right\} . \tag{4.2.1}$$

For the polar coordinate system in a plane:

$$ds^2 = d\rho^2 + \rho^2\, d\theta^2$$

$$\left.\begin{aligned} E &= 1 \\ G &= \rho^2 \end{aligned}\right\} . \tag{4.2.2}$$

Both of the systems are orthogonal, so

$$f = F = 0 . \tag{4.2.3}$$

The origin of the projection has coordinates (ϕ_0, λ_0), λ_0 being some longitude on the parallel of tangency. Coordinate λ_0 defines the central meridian of the map.

Impose the conditions that

$$\rho = \rho(\phi) \tag{4.2.4}$$

$$\theta = c_1\lambda + c_2 . \tag{4.2.5}$$

The constant c_2 is zero, if a further condition is that $\lambda = 0$, when $\theta = 0$.

The condition of equivalency of area is:

$$eg - f^2 = \begin{vmatrix} E & F \\ F & G \end{vmatrix} \begin{vmatrix} \frac{\partial \rho}{\partial \phi} & \frac{\partial \rho}{\partial \lambda} \\ \frac{\partial \theta}{\partial \phi} & \frac{\partial \theta}{\partial \lambda} \end{vmatrix}^2 . \tag{4.2.6}$$

From (4.2.4)

$$\frac{\partial \rho}{\partial \lambda} = 0 . \tag{4.2.7}$$

From (4.2.5)

$$\frac{\partial \theta}{\partial \phi} = 0 \tag{4.2.8}$$

$$\frac{\partial \theta}{\partial \lambda} = c_1 . \tag{4.2.9}$$

Substitute (4.2.1), (4.2.2), (4.2.3), (4.2.7), (4.2.8), and (4.2.9) into (4.2.6).

$$R^4 \cos^2 \phi = \rho^2 \begin{vmatrix} \frac{\partial \rho}{\partial \phi} & 0 \\ 0 & c_1 \end{vmatrix}^2$$

$$R^2 \cos \phi = -\rho c_1 \left(\frac{\partial \rho}{\partial \phi} \right) \tag{4.2.10}$$

The minus sign is chosen since an increase in ϕ corresponds to a decrease in ρ.

Convert (4.2.10) into an ordinary differential equation, and integrate.

$$\rho \, d\rho = -\frac{R^2}{c_1} \cos \phi \, d\phi$$

$$\rho^2 = -\frac{2R^2}{c_1} \sin \phi + c_3 . \tag{4.2.11}$$

In (4.2.11), c_1 will become the constant of the cone, as is shown below, and c_3 will depend on the boundary conditions imposed.

The plane Cartesian coordinates of the map are given by:

$$x = \rho \sin \theta$$

$$y = \rho_0 - \rho \cos \theta \, . \qquad (4.2.12)$$

This development will now be applied to the Conical Equal Area projection with one standard parallel, which is also called Albers' projection. From Chapter 1,

$$\rho_0 = R \cot \phi_0 \qquad (4.2.13)$$

$$\theta = \lambda \sin \phi_0 \, . \qquad (4.2.14)$$

Comparing (4.2.5) and (4.2.14),

$$c_1 = \sin \phi_0 \, . \qquad (4.2.15)$$

The constant c_1 is the constant of the cone (of Chapter 1). Substitute (4.2.15) into (4.2.11).

$$\rho^2 = -2R^2 \frac{\sin \phi}{\sin \phi_0} + c_3 \, . \qquad (4.2.16)$$

Evaluate (4.2.16) at ϕ_0.

$$\rho_0^2 = -2R^2 + c_3$$

$$c_3 = \rho_0^2 + 2R^2 \, . \qquad (4.2.17)$$

Substitute (4.2.13) into (4.2.17).

$$c_3 = R^2 \cot^2 \phi_0 + 2R^2$$

$$= R^2 (2 + \cot^2 \phi_0) \, . \qquad (4.2.18)$$

Substitute (4.2.18) into (4.2.16).

$$\rho^2 = R^2 \left(2 + \cot^2 \phi_0 - 2 \frac{\sin \phi}{\sin \phi_0} \right)$$

$$= R^2 \left(2 \frac{\sin^2 \phi_0}{\sin^2 \phi_0} + \frac{\cos^2 \phi_0}{\sin^2 \phi_0} - 2 \frac{\sin \phi \sin \phi_0}{\sin^2 \phi_0} \right)$$

$$= \frac{R^2}{\sin^2 \phi_0} (1 + \sin^2 \phi_0 - 2 \sin \phi \sin \phi_0)$$

$$\rho = \frac{R}{\sin\phi_0}\sqrt{1 + \sin^2\phi_0 - 2\sin\phi\sin\phi_0} \,. \tag{4.2.19}$$

The Cartesian plotting equations have been derived. From (4.2.12), (4.2.13), and (4.2.14), and including (4.2.19)

$$x = S[\rho \sin(\lambda \sin\phi_0)]$$

$$y = S[R\cot\phi_0 - \rho\cos(\lambda\sin\phi_0)] \tag{4.2.20}$$

where S is the scale factor.

Equations (4.2.19) and (4.2.20) are the basis of the plotting Table 4.2.1, for $\phi_0 = 45°$ and $\lambda_0 = 0°$. No generalized grid is possible; each map is the choice of the user. The grid that resulted for this arbitrary choice is Figure 4.2.1.

Two standard parallels may also be selected for a secant cone. This projection is also called the Albers projection. The radii on the map for the standard parallels is:

$$\rho_1 c_1 = R\cos\phi_1 \tag{4.2.21}$$

$$\rho_2 c_1 = R\cos\phi_2 \,. \tag{4.2.22}$$

From (4.2.5)

$$\theta = c_1\lambda = \frac{\lambda R\cos\phi_1}{\rho_1} = \frac{\lambda R\cos\phi_2}{\rho_2} \,. \tag{4.2.23}$$

Substituting (4.2.21) and (4.2.22) into (4.2.11) yields

$$\frac{R^2\cos^2\phi_1}{c_1^2} = -\frac{2R^2}{c_1}\sin\phi_1 + c_3$$

$$R^2\cos^2\phi_1 + 2R^2 c_1 \sin\phi_1 - c_1^2 c_3 = 0 \tag{4.2.24}$$

$$\frac{R^2\cos^2\phi_2}{c_1^2} = -\frac{2R^2}{c_1}\sin\phi_2 + c_3$$

$$R^2\cos^2\phi_2 + 2R^2 c_1 \sin\phi_2 - c_1^2 c_3 = 0 \,. \tag{4.2.25}$$

Table 4.2.1. Conical Equal Area Projection, One Standard Parallel.

Equal Area Conical One Standard Parallel

Latitude*	Longitude*	X**	Y**
0.0000	0.0000	0.000	-[illegible].669
0.0000	15.0800	2.033	-4.480
0.0000	30.0000	3.997	-3.921
0.0000	45.0000	5.825	-3.009
0.0000	60.0000	7.453	-1.776
0.0000	75.0000	8.827	-.265
0.0000	90.0000	9.699	1.473
15.0000	0.0000	0.000	-3.227
15.0000	15.0000	1.769	-3.063
15.0000	30.0000	3.476	-2.576
15.0000	45.0000	5.064	-1.784
15.0000	60.0000	6.480	-.712
15.0000	75.0000	7.675	.602
15.0000	90.0000	8.607	2.113
30.0000	0.0000	0.000	-1.654
30.0000	15.0000	1.479	-1.517
30.0000	30.0000	2.906	-1.109
30.0000	45.0000	4.235	-.447
30.0000	60.0000	5.419	.450
30.0000	75.0000	6.417	1.948
30.0000	90.0000	7.197	2.812
45.0000	0.0000	0.000	.000
45.0000	15.0000	1.174	.109
45.0000	30.0000	2.309	.432
45.0000	45.0000	3.363	.959
45.0000	60.0000	4.303	1.670
45.0000	75.0000	5.096	2.543
45.0000	90.0000	5.715	3.546
60.0000	0.0000	0.000	1.646
60.0000	15.0000	.871	1.727
60.0000	30.0000	1.712	1.966
60.0000	45.0000	2.495	2.357
60.0000	60.0000	3.193	2.885
60.0000	75.0000	3.781	3.532
60.0000	90.0000	4.240	4.277

$\phi_0 = 45^\circ$ *Degrees

$\lambda_0 = 0^\circ$ **Meters

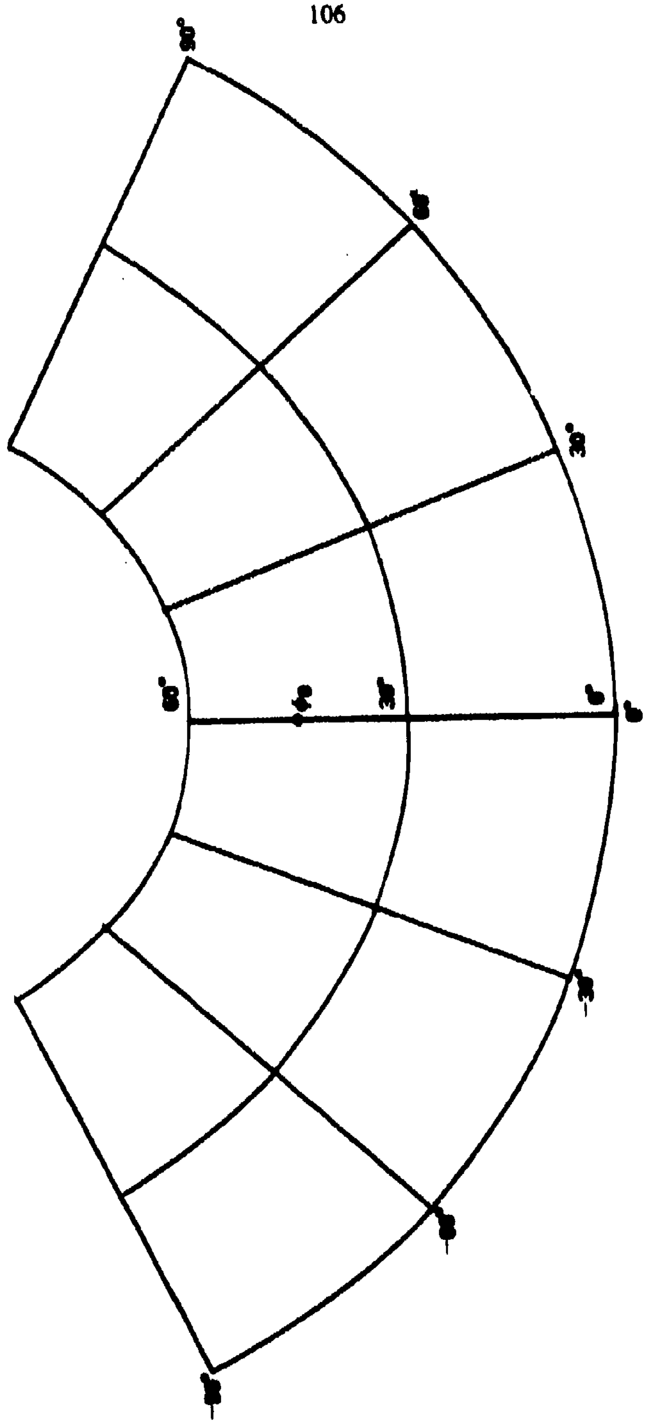


Figure 4.2.1. Conical equal area projection, one standard parallel

Equations (4.2.24) and (4.2.25) can be solved simultaneously for c_1.

$$R^2 \cos^2\phi_1 - R^2 \cos^2\phi_2 + 2R^2 c_1 \sin\phi_1 - 2R^2 c_1 \sin\phi_2 = 0$$

$$\cos^2\phi_1 - \cos^2\phi_2 = 2c_1 (\sin\phi_2 - \sin\phi_1)$$

$$c_1 = \frac{\cos^2\phi_1 - \cos^2\phi_2}{2(\sin\phi_2 - \sin\phi_1)}$$

$$= \frac{\sin^2\phi_2 - \sin^2\phi_1}{2(\sin\phi_2 - \sin\phi_1)}$$

$$= \frac{1}{2} (\sin\phi_2 + \sin\phi_1)\,. \tag{4.2.26}$$

Substituting (4.2.26) into (4.2.23)

$$\theta = \frac{\lambda}{2} (\sin\phi_1 + \sin\phi_2)\,. \tag{4.2.27}$$

Substitute (4.2.26) into (4.2.11)

$$\rho^2 = \frac{4R^2 \sin\phi}{\sin\phi_1 + \sin\phi_2} + c_3\,. \tag{4.2.28}$$

Evaluate (4.2.28) at ϕ_1.

$$\rho_1^2 = -\frac{4R^2 \sin\phi_1}{\sin\phi_1 + \sin\phi_2} + c_3$$

$$c_3 = \rho_1^2 + \frac{4R^2 \sin\phi_1}{\sin\phi_1 + \sin\phi_2}\,. \tag{4.2.29}$$

Substitute (4.2.29) into (4.2.28)

$$\rho^2 = \rho_1^2 + 4R^2 \frac{(\sin\phi_1 - \sin\phi)}{\sin\phi_1 + \sin\phi_2} \tag{4.2.30}$$

where, from (4.2.21) and (4.2.26),

$$\rho_1 = \frac{2R\cos\phi_1}{\sin\phi_1 + \sin\phi_2}\,. \tag{4.2.31}$$

A similar development will give

$$\rho^2 = \rho_2^2 + 4R^2 \frac{(\sin\phi_2 - \sin\phi)}{\sin\phi_1 + \sin\phi_2} \tag{4.2.32}$$

$$\rho_2 = \frac{2R\cos\phi_2}{\sin\phi_1 + \sin\phi_2}\,. \tag{4.2.33}$$

It only remains to substitute into (4.2.12) to obtain the plotting equations. One form of these is

$$x = S \cdot \sqrt{\rho_1^2 + 4R^2 \frac{(\sin\phi_1 - \sin\phi)}{\sin\phi_1 + \sin\phi_2}}\ \sin\left[\frac{\lambda}{2}(\sin\phi_1 + \sin\phi_2)\right] \tag{4.2.34}$$

$$y = S \cdot \left\{\frac{1}{2}(\rho_1 + \rho_2) - \sqrt{\rho_1^2 + 4R^2 \frac{(\sin\phi_1 - \sin\phi)}{\sin\phi_1 + \sin\phi_2}}\ \cos\left[\frac{\lambda}{2}(\sin\phi_1 + \sin\phi_2)\right]\right\} \tag{4.2.35}$$

where S is the scale factor.

The second form is

$$x = S\sqrt{\rho_2^2 + \frac{4R^2(\sin\phi_2 - \sin\phi)}{\sin\phi_1 + \sin\phi_2}}\ \sin\left[\frac{\lambda}{2}(\sin\phi_1 + \sin\phi_2)\right] \tag{4.2.36}$$

$$y = S\left\{\frac{1}{2}(\rho_1 + \rho_2) - \sqrt{\rho_2^2 + \frac{4R^2(\sin\phi_2 - \sin\phi)}{\sin\phi_1 + \sin\phi_2}}\ \cos\left[\frac{\lambda}{2}(\sin\phi_1 + \sin\phi_2)\right]\right\}. \tag{4.2.37}$$

The grid is shown in Figure 4.2.2 for the two standard parallel case. The plotting table for the grid is Table 4.2.2.

The standard parallels were chosen as $\phi_1 = 30°$, and $\phi_2 = 60°$. The central meridian is $\lambda_0 = 0°$.

A second approach can be followed for the single standard parallel case.

Consider a cone with constant $\sin\phi_0$, and let ρ_0 be the radius on the map of the standard parallel ϕ_0. The area on the cone bounded by that parallel is $\pi\rho_0^2 \sin\phi_0$. If ρ is the radius of any other parallel of latitude ϕ, then the area bounded is $\pi\rho^2 \sin\phi$. The area in the strip between these parallels is

$$A = \pi(\rho_0^2 - \rho^2)\sin\phi_0\,. \tag{4.2.38}$$

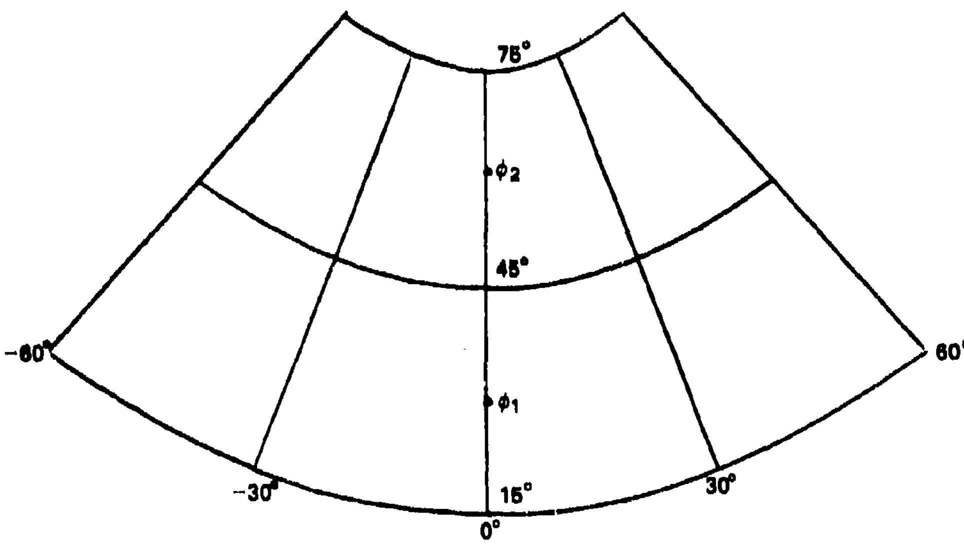


Figure 4.2.2. Conical equal area projection, two standard parallels

**Table 4.2.2. Conical Equal Area Projection,
Two Standard Parallels.**

Equal Area Conical Two Standard Parallels

Latitude*	Longitude*	X**	Y**
15.0000	0.0000	0.000	-3.324
15.0000	15.0000	1.726	-3.169
15.0000	30.0000	3.396	-2.710
15.0000	45.0000	4.959	-1.961
15.0000	60.0000	6.363	-.946
30.0000	0.0000	0.000	-1.709
30.0000	15.0000	1.438	-1.580
30.0000	30.0000	2.831	-1.197
30.0000	45.0000	4.133	-.573
30.0000	60.0000	5.304	.273
45.0000	0.0000	0.000	-.004
45.0000	15.0000	1.135	.098
45.0000	30.0000	2.234	.400
45.0000	45.0000	3.262	.893
45.0000	60.0000	4.185	1.560
60.0000	0.0000	0.000	1.709
60.0000	15.0000	.830	1.783
60.0000	30.0000	1.634	2.004
60.0000	45.0000	2.386	2.365
60.0000	60.0000	3.062	2.853
75.0000	0.0000	0.000	3.232
75.0000	15.0000	.560	3.282
75.0000	30.0000	1.101	3.431
75.0000	45.0000	1.608	3.674
75.0000	60.0000	2.063	4.063

$\phi_1 = 30°$ *Degrees
$\phi_2 = 60°$ **Meters
$\lambda_0 = 0°$

The area on the authalic sphere between the parallels ϕ_0 and ϕ is

$$A = 2\pi R^2 (\sin\phi - \sin\phi_0) . \tag{4.2.39}$$

For equal area, equate (4.2.38) and (4.2.39).

$$\pi(\rho_0^2 - \rho^2) \sin\phi_0 = 2\pi R^2 (\sin\phi - \sin\phi_0)$$

$$(\rho_0^2 - \rho^2) \sin\phi_0 = 2R^2 (\sin\phi - \sin\phi_0) . \tag{4.2.40}$$

Substitute (4.2.13) into (4.2.40).

$$\sin\phi_0 (R^2 \cot^2\phi_0 - \rho^2) = 2R^2 (\sin\phi - \sin\phi_0)$$

$$\therefore \; R^2 \cot^2\phi_0 - \rho^2 = 2R^2 \frac{\sin\phi}{\sin\phi_0} - 2R^2$$

$$\rho^2 = R^2 \cot^2\phi_0 + 2R^2 - 2R^2 \frac{\sin\phi}{\sin\phi_0} . \tag{4.2.41}$$

Equation (4.2.41) is the equivalent of (4.2.19).

Note the difference in the length of the derivations between the first and second approaches. While the method of differential geometry seems more tedious, it will pay dividends in Chapter 7. The equations for a quantitative estimate of distortion will be seen to follow from the differential geometry approach. The second means of derivation leaves the reader without a convenient way of exploring distortions.

Similarly, the case of Albers projection with two standard parallels can be handled with an alternate approach.

Let ϕ_1 and ϕ_2 be the latitude of the two standard parallels, and ρ_1 and ρ_2 be their respective radii on the projection. Let ϕ_2 be greater than ϕ_1. The constant of the cone is c.

The area of the strip of the cone between these latitudes is

$$A = c\pi(\rho_1^2 - \rho_2^2) . \tag{4.2.42}$$

The area of a zone of the authalic sphere between the given latitudes is

$$A = 2\pi R^2 (\sin\phi_2 - \sin\phi_1) . \tag{4.2.43}$$

For the equal area projection, equate (4.2.42) and (4.2.43).

$$c\pi(\rho_1^2 - \rho_2^2) = 2\pi R^2 (\sin\phi_2 - \sin\phi_1)$$

$$c(\rho_1^2 - \rho_2^2) = 2R^2 (\sin\phi_2 - \sin\phi_1) . \tag{4.2.44}$$

Since the standard parallels are true length, equate these parallels on the map, and on the authalic sphere.

$$2\pi\rho_1 c = 2\pi R \cos\phi_1, \quad \rho_1 = \frac{R\cos\phi_1}{c} \tag{4.2.45}$$

$$2\pi\rho_2 c = 2\pi R \cos\phi_2, \quad \rho_2 = \frac{R\cos\phi_2}{c} \tag{4.2.46}$$

Substitute (4.2.45) and (4.2.46) into (4.2.44).

$$c\left(\frac{R^2\cos^2\phi_1}{c^2} - \frac{R^2\cos^2\phi_2}{c}\right) = 2R^2(\sin\phi_2 - \sin\phi_1)$$

$$\frac{\cos^2\phi_1 - \cos^2\phi_2}{c} = 2(\sin\phi_2 - \sin\phi_1)$$

$$c = \frac{\cos^2\phi_1 - \cos^2\phi_2}{2(\sin\phi_2 - \sin\phi_1)}$$

$$= \frac{\sin^2\phi_2 - \sin^2\phi_1}{2(\sin\phi_2 - \sin\phi_1)}$$

$$= \frac{1}{2}(\sin\phi_1 + \sin\phi_2).$$

We have now reproduced equation (4.2.27)

Substituting the constant of the cone into (4.2.45) and (4.2.46)

$$\rho_1 = \frac{2R\cos\phi_1}{\sin\phi_1 + \sin\phi_2}$$

$$\rho_2 = \frac{2R\cos\phi_2}{\sin\phi_1 + \sin\phi_2}.$$

This has reproduced (4.2.31) and (4.2.33)

To find the value of ρ for a general latitude ϕ, again equate the area on the map and the area on the authalic sphere.

$$c\pi(\rho^2 - \rho_1^2) = 2\pi R^2(\sin\phi_1 - \sin\phi)$$

$$\rho^2 = \rho_1^2 + 2R^2(\sin\phi_1 - \sin\phi)$$

$$= \rho_1^2 + \frac{4R^2(\sin\phi_1 - \sin\phi)}{\sin\phi_1 + \sin\phi_2}.$$

This reproduces (4.2.30).

The Albers projection has been used extensively in geographic atlases to portray areas of large east to west extent. The two standard parallel case has been successfully used for maps of the United States, since distortion is a function of latitude, and not longitude. An estimate of the distortion inherent in this projection is given in Chapter 7.

4.3 Azimuthal Projections [2], [20], [22]

The Azimuthal Equal Area projections may be polar, equatorial, or oblique projections of the authalic sphere directly onto a plane. Two methods of derivation of the polar case will be considered. Then, it will be noted how the oblique and equatorial cases can be obtained by the rotational transformations of Chapter 2.

The Azimuthal Equal Area polar projection (also called the Lambert Azimuthal Equivalent projection) can easily obtained from the Conical projection with one standard parallel by setting $\phi_0 = 90°$ in equations (4.2.14) and (4.2.17).

$$\theta = \lambda \tag{4.3.1}$$

$$\rho = R\sqrt{2(1 - \sin\phi)}\,. \tag{4.3.2}$$

The plotting equations, in Cartesian coordinates, are, including the scale factor, S,

$$x = R \cdot S\sqrt{2(1 - \sin\phi)}\,\sin\lambda \tag{4.3.3}$$

$$y = -R \cdot S\sqrt{2(1 - \sin\phi)}\,\cos\lambda\,. \tag{4.3.4}$$

The result of plotting these formulas (plotting Table 4.3.1) is Figure 4.3.1. The parallels are concentric circles, unevenly spaced, and the meridians are straight lines. The distortion becomes severe as the equator is reached.

The second way to derive the polar case is equally brief. The area of the segment of the authalic sphere surrounding the pole, and above the latitude ϕ is

$$\begin{aligned} A &= 2\pi R(R - R\sin\phi) \\ &= 2\pi R^2(1 - \sin\phi)\,. \end{aligned} \tag{4.3.5}$$

This will be transformed into a circle of radius ρ with area

$$A = \pi\rho^2\,. \tag{4.3.6}$$

Equating (4.3.5) and (4.3.6)

$$\pi\rho^2 = 2\pi R^2(1 - \sin\phi)$$

$$\rho = R\sqrt{2(1 - \sin\phi)}\,.$$

Equation (4.3.2) has been duplicated.

Table 4.3.1. Azimuthal Equal Area Projection, Polar Case.

Equal Area Azimuthal

Latitude*	Longitude*	X**	Y**
0.0000	0.0000	9.020	0.000
0.0000	15.0000	8.713	2.335
0.0000	30.0000	7.812	4.510
0.0000	45.0000	6.378	6.378
0.0000	60.0000	4.510	7.812
0.0000	75.0000	2.334	8.713
0.0000	90.0000	-.000	9.020
15.0000	0.0000	7.766	0.000
15.0000	15.0000	7.501	2.010
15.0000	30.0000	6.725	3.883
15.0000	45.0000	5.491	5.491
15.0000	60.0000	3.883	6.725
15.0000	75.0000	2.010	7.501
15.0000	90.0000	-.000	7.766
30.0000	0.0000	6.378	0.000
30.0000	15.0000	6.161	1.651
30.0000	30.0000	5.524	3.189
30.0000	45.0000	4.510	4.510
30.0000	60.0000	3.189	5.524
30.0000	75.0000	1.651	6.161
30.0000	90.0000	-.000	6.378
45.0000	0.0000	4.882	0.000
45.0000	15.0000	4.715	1.263
45.0000	30.0000	4.228	2.441
45.0000	45.0000	3.452	3.452
45.0000	60.0000	2.441	4.228
45.0000	75.0000	1.263	4.715
45.0000	90.0000	-.000	4.882
60.0000	0.0000	3.301	0.000
60.0000	15.0000	3.189	.854
60.0000	30.0000	2.859	1.651
60.0000	45.0000	2.334	2.335
60.0000	60.0000	1.651	2.859
60.0000	75.0000	.854	3.189
60.0000	90.0000	-.000	3.301
75.0000	0.0000	1.665	0.000
75.0000	15.0000	1.608	.431
75.0000	30.0000	1.442	.832
75.0000	45.0000	1.177	1.177
75.0000	60.0000	.832	1.442
75.0000	75.0000	.431	1.608
75.0000	90.0000	-.000	1.665
90.0000	0.0000	.000	0.000
90.0000	15.0000	.000	.000
90.0000	30.0000	.000	.000
90.0000	45.0000	.000	.000
90.0000	60.0000	.000	.000
90.0000	75.0000	.000	.000
90.0000	90.0000	-.000	.000

$\phi_0 = 90^\circ$ $\lambda_0 = 0^\circ$ *Degrees **Meters

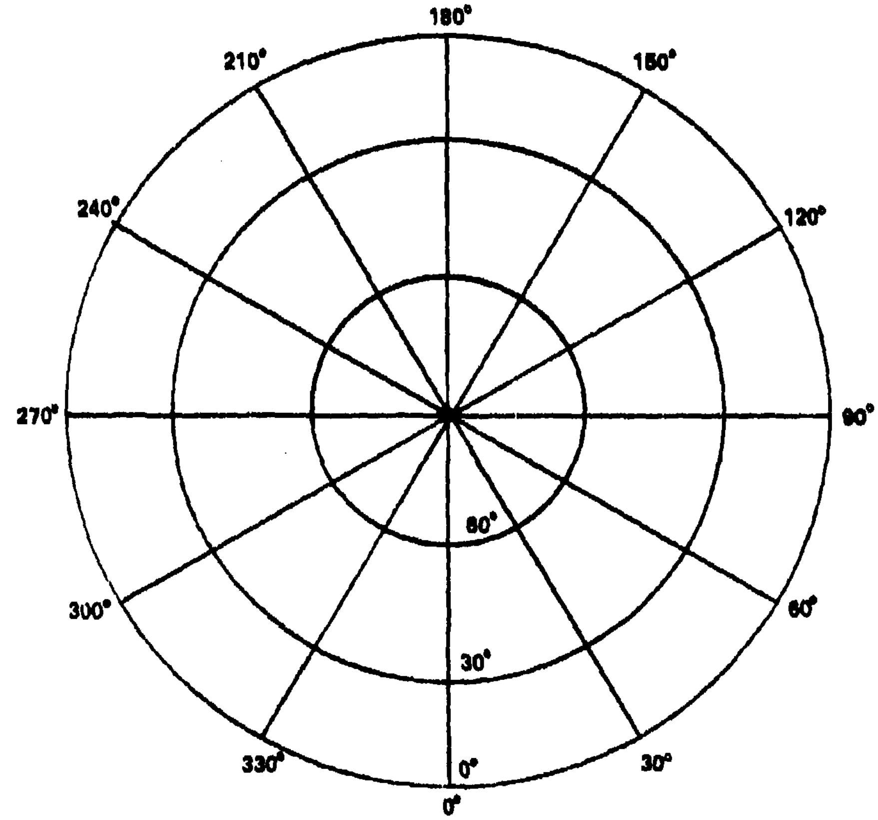


Figure 4.3.1. Azimuthal equal area projection, polar case

The oblique and equatorial cases can be obtained from the polar case by applying the rotations of Section 2.10. This has been done in generating the plotting of Tables 4.3.2 and 4.3.3, and Figures 4.3.2 and 4.3.3. For Figure 4.3.3, the point of tangency of the plane against the sphere was arbitrarily chosen at 45° north latitude.

To obtain the oblique variation, write (4.3.3) and (4.3.4) in the auxiliary coordinate system.

$$x = R \cdot S \sqrt{2(1 - \sin h)} \sin \alpha \tag{4.3.7}$$

$$y = R \cdot S \sqrt{2(1 - \sin h)} \cos \alpha . \tag{4.3.8}$$

From (2.10.4) and (2.10.5)

$$\sin h = \sin \phi \sin \phi_p + \cos \phi \cos \phi_p \cos \lambda \tag{4.3.9}$$

$$\tan \alpha = \frac{\sin \lambda}{\cos \phi_p \tan \phi - \sin \phi_p \cos \lambda}$$

$$\alpha = \tan^{-1} \left(\frac{\sin \lambda}{\cos \phi_p \tan \phi - \sin \phi_p \cos \lambda} \right). \tag{4.3.10}$$

Substitute (4.3.9) and (4.3.10) into (4.3.7) and (4.3.8).

$$\left.\begin{aligned} x &= R \cdot S \sqrt{2(1 - \sin \phi \sin \phi_p - \cos \phi \cos \phi_p \cos \lambda)} \\ &\quad \times \sin \left[\tan^{-1} \left(\frac{\sin \lambda}{\cos \phi_p \tan \phi - \sin \phi_p \cos \lambda} \right) \right] \\ y &= R \cdot S \sqrt{2(1 - \sin \phi \sin \phi_p - \cos \phi \cos \phi_p \cos \lambda)} \\ &\quad \times \cos \left[\tan^{-1} \left(\frac{\sin \lambda}{\cos \phi_p \tan \phi - \sin \phi_p \cos \lambda} \right) \right] \end{aligned}\right\} . \tag{4.3.11}$$

To obtain the equatorial azimuthal equal area projection, substitute $\phi_0 = 0°$ into (4.3.11)

$$\left.\begin{aligned} x &= R \cdot S \cdot \sqrt{2(1 - \cos \phi \cos \lambda)} \sin \left[\tan^{-1} \left(\frac{\sin \lambda}{\tan \phi} \right) \right] \\ y &= R \cdot S \cdot \sqrt{2(1 - \cos \phi \cos \lambda)} \cos \left[\tan^{-1} \left(\frac{\sin \lambda}{\tan \phi} \right) \right] \end{aligned}\right\} . \tag{4.3.12}$$

The formulas for the distortion in the polar case are given in Chapter 7.

Table 4.3.2. Azimuthal Equal Area Projection, Oblique Case.

Equal Area Azimuthal, Oblique Case

Latitude*	Longitude*	X**	Y**
0.0000	0.0000	-.000	-4.882
0.0000	30.0000	3.552	-4.350
0.0000	60.0000	6.714	-2.741
0.0000	90.0000	9.020	.000
30.0000	0.0000	-.000	-1.665
30.0000	30.0000	2.846	-1.162
30.0000	60.0000	5.251	.332
30.0000	90.0000	6.714	2.741
60.0000	0.0000	0.000	1.665
60.0000	30.0000	1.628	1.994
60.0000	60.0000	2.920	2.937
60.0000	90.0000	3.552	4.350
90.0000	0.0000	-.000	4.882
90.0000	30.0000	.000	4.882
90.0000	60.0000	.000	4.882
90.0000	90.0000	.000	4.882

$\phi_0 = 45°$ *Degrees
$\lambda_0 = 0°$ **Meters

Table 4.3.3. Azimuthal Equal Area Projection, Equatorial Case.

Equal Area Azimuthal, Equatorial Case

Latitude*	Longitude*	X**	Y**
0.0000	0.0000	.000	.000
0.0000	15.0000	1.665	.000
0.0000	30.0000	3.302	.000
0.0000	45.0000	4.882	.000
0.0000	60.0000	6.378	.000
0.0000	75.0000	7.766	.000
0.0000	90.0000	9.020	.000
15.0000	0.0000	0.000	1.665
15.0000	15.0000	1.622	1.679
15.0000	30.0000	3.215	1.723
15.0000	45.0000	4.749	1.800
15.0000	60.0000	6.196	1.917
15.0000	75.0000	7.527	2.088
15.0000	90.0000	8.713	2.335
30.0000	0.0000	0.000	3.302
30.0000	15.0000	1.492	3.328
30.0000	30.0000	2.953	3.409
30.0000	45.0000	4.350	3.552
30.0000	60.0000	5.651	3.768
30.0000	75.0000	6.820	4.076
30.0000	90.0000	7.812	4.510
45.0000	0.0000	0.000	4.882
45.0000	15.0000	1.272	4.916
45.0000	30.0000	2.511	5.023
45.0000	45.0000	3.682	5.208
45.0000	60.0000	4.748	5.482
45.0000	75.0000	5.664	5.864
45.0000	90.0000	6.378	6.378
60.0000	0.0000	0.000	6.378
60.0000	15.0000	.959	6.415
60.0000	30.0000	1.884	6.526
60.0000	45.0000	2.741	6.714
60.0000	60.0000	3.493	6.987
60.0000	75.0000	4.099	7.351
60.0000	90.0000	4.510	7.812
75.0000	0.0000	0.000	7.766
75.0000	15.0000	.540	7.793
75.0000	30.0000	1.055	7.875
75.0000	45.0000	1.518	8.011
75.0000	60.0000	1.902	8.198
75.0000	75.0000	2.183	8.435
75.0000	90.0000	2.334	8.713
90.0000	0.0000	-.000	9.020
90.0000	15.0000	.000	9.020
90.0000	30.0000	.000	9.020
90.0000	45.0000	.000	9.020
90.0000	60.0000	.000	9.020
90.0000	75.0000	.000	9.020
90.0000	90.0000	.000	9.020

$\phi_0 = 0^\circ$ $\lambda_0 = 0^\circ$ *Degrees **Meters

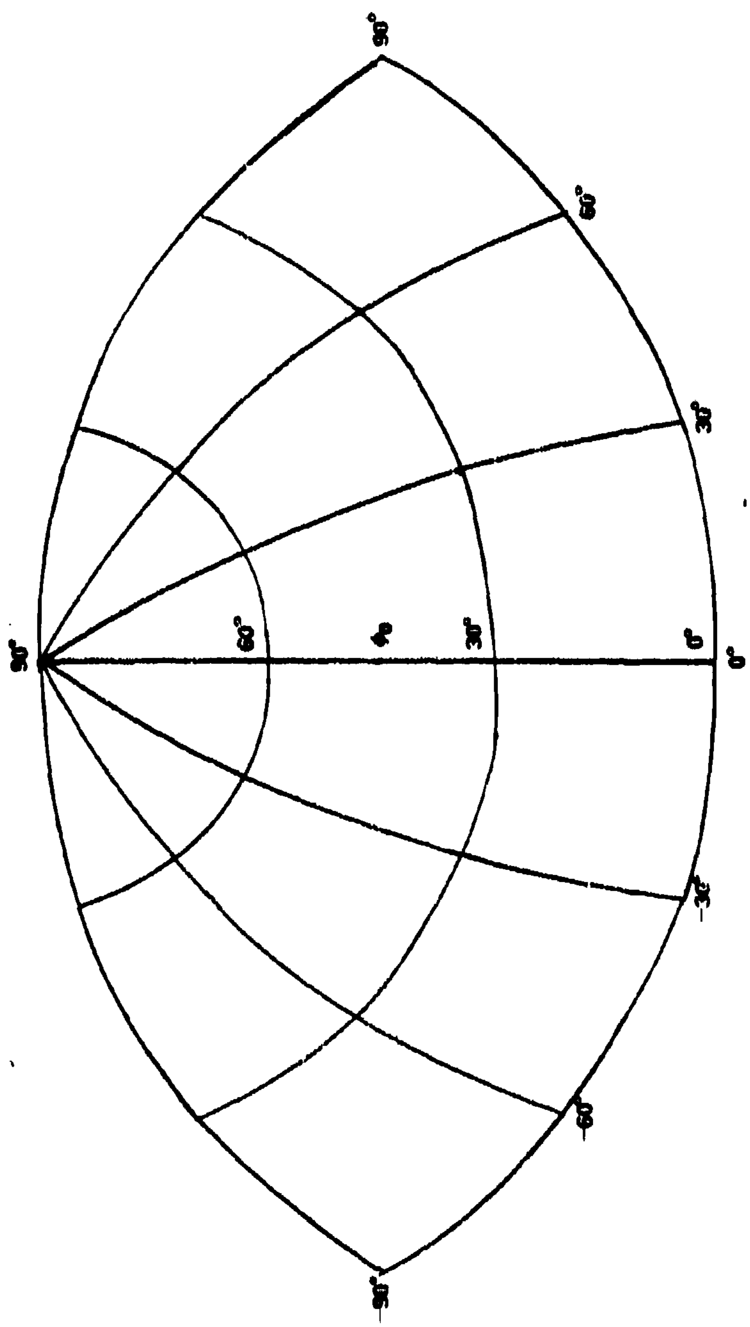


Figure 4.3.2 Azimuthal equal area, oblique case

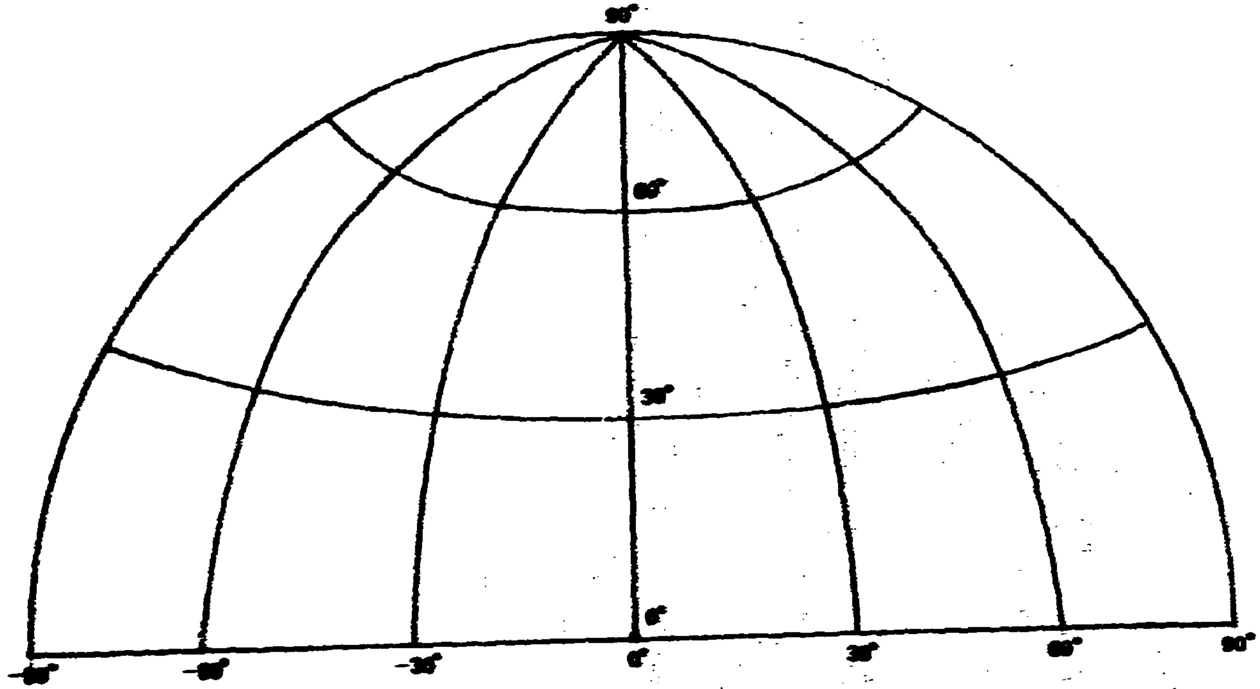


Figure 4.1.3. Azimuthal equal area, equatorial case

4.4 Bonne's Projection [2], [8], [20]

Bonne's projection is a modified conical equal area projection. The only straight line in the map is the central meridian, which is a section of the cone and the central meridianal plane.

On the authalic sphere,

$$ds^2 = R^2 \, d\phi^2 + R^2 \cos^2\phi \, d\lambda^2$$

$$\left.\begin{aligned} e &= R^2 \\ g &= R^2 \cos^2\phi \end{aligned}\right\} . \tag{4.4.1}$$

In the plane, a polar coordinate system has

$$ds^2 = d\rho^2 + \rho^2 \, d\theta^2$$

$$\left.\begin{aligned} E &= 1 \\ G &= \rho^2 \end{aligned}\right\} . \tag{4.4.2}$$

The condition for equivalency of area is

$$eg - f^2 = \begin{vmatrix} E & F \\ F & G \end{vmatrix} \begin{vmatrix} \dfrac{\partial\rho}{\partial\phi} & \dfrac{\partial\rho}{\partial\lambda} \\ \dfrac{\partial\theta}{\partial\phi} & \dfrac{\partial\theta}{\partial\lambda} \end{vmatrix}^2 . \tag{4.4.3}$$

On the sphere, $f = 0$. On the plane, $F = 0$ on the central meridian. Substituting these into (4.4.3).

$$eg = EG \begin{vmatrix} \dfrac{\partial\rho}{\partial\phi} & \dfrac{\partial\rho}{\partial\lambda} \\ \dfrac{\partial\theta}{\partial\phi} & \dfrac{\partial\theta}{\partial\lambda} \end{vmatrix}^2 . \tag{4.4.4}$$

The radius of the parallel circle of latitude ϕ is

$$\rho = \rho_0 - \int_{\phi_0}^{\phi} R \, d\phi . \tag{4.4.5}$$

From (4.4.5)

$$\left.\begin{aligned} \frac{\partial\rho}{\partial\phi} &= -R \\ \frac{\partial\rho}{\partial\lambda} &= 0 \end{aligned}\right\} , \tag{4.4.6}$$

Substituting (4.4.1), (4.4.2), and (4.4.6) into (4.4.4).

$$R^4 \cos^2\phi = \rho^2 \begin{vmatrix} -R & 0 \\ \frac{\partial\theta}{\partial\rho} & \frac{\partial\theta}{\partial\lambda} \end{vmatrix}^2$$

$$= R^2\rho^2 \left(\frac{\partial\theta}{\partial\lambda}\right)^2$$

$$R \cos\phi = \rho \left(\frac{\partial\theta}{\partial\lambda}\right). \tag{4.4.7}$$

Convert (4.4.7) into an ordinary differential equation, and integrate.

$$R \cos\phi \, d\lambda = \rho \, d\theta$$

$$\lambda R \cos\phi = \rho\theta - c'$$

$$\theta = \frac{\lambda R \cos\phi}{\rho} + c .$$

The constant c is zero if the condition is imposed that $\lambda = 0$ when $\theta = 0$.

$$\theta = \frac{\lambda R \cos\phi}{\rho}. \tag{4.4.8}$$

In Chapter 1, it was noted that for a conical projection, tangent to a sphere,

$$\rho_0 = R \cot\phi_0 . \tag{4.4.9}$$

Carrying out the integration of (4.4.5)

$$\rho = \rho_0 - R(\phi - \phi_0) . \tag{4.4.10}$$

Substitute (4.4.9) into (4.4.10).

$$\rho = R \cot \phi_0 - R(\phi - \phi_0) . \tag{4.4.11}$$

The Cartesian plotting coordinates follow simply, with y-axis as the central meridian, and the origin corresponding to ϕ_0, from (4.4.8) and (4.4.11).

$$x = \rho S \sin\left(\frac{\lambda R \cos\phi}{\rho}\right) \tag{4.4.12}$$

$$y = S\left[R \cot\phi_0 - \rho \cos\left(\frac{\lambda R \cos\phi}{\rho}\right)\right] \tag{4.4.13}$$

where S is the scale factor and λ is in radians.

The Bonne projection cannot be generalized. It applies to a specific case, with a specified standard parallel, ϕ_0. In order to demonstrate the projection, a plotting table for $\phi_0 = 45$ has been placed in Table 4.4.1.

The Bonne projection has been used as a military map by France.

The grid itself is given in Figure 4.4.1. Note the curvature of the meridians, and the fact that the parallels of latitude are concentric circles.

Table 4.4.1. Bonne Equal Area Projection.

Bonne

Latitude*	Longitude*	X**	Y**
0.0000	0.0000	0.000	-5.009
0.0000	15.0000	1.664	-4.887
0.0000	30.0000	3.292	-4.523
0.0000	45.0000	4.849	-3.925
0.0000	60.0000	6.303	-3.106
0.0000	75.0000	7.621	-2.083
0.0000	90.0000	8.775	-.879
15.0000	0.0000	0.000	-3.340
15.0000	15.0000	1.606	-3.206
15.0000	30.0000	3.167	-2.809
15.0000	45.0000	4.641	-2.160
15.0000	60.0000	5.988	-1.275
15.0000	75.0000	7.170	-.181
15.0000	90.0000	8.155	1.094
30.0000	0.0000	0.000	-1.670
30.0000	15.0000	1.438	-1.540
30.0000	30.0000	2.830	-1.156
30.0000	45.0000	4.131	-.529
30.0000	60.0000	5.299	.321
30.0000	75.0000	6.296	1.366
30.0000	90.0000	7.091	2.572
45.0000	0.0000	0.000	.000
45.0000	15.0000	1.174	.109
45.0000	30.0000	2.308	.432
45.0000	45.0000	3.363	.959
45.0000	60.0000	4.303	1.670
45.0000	75.0000	5.096	2.543
45.0000	90.0000	5.715	3.546
60.0000	0.0000	0.000	1.670
60.0000	15.0000	.831	1.744
60.0000	30.0000	1.635	1.963
60.0000	45.0000	2.388	2.320
60.0000	60.0000	3.066	2.805
60.0000	75.0000	3.649	3.402
60.0000	90.0000	4.116	4.093

$\phi_0 = 45^\circ$ *Degrees
$\lambda_0 = 0^\circ$ **Meters

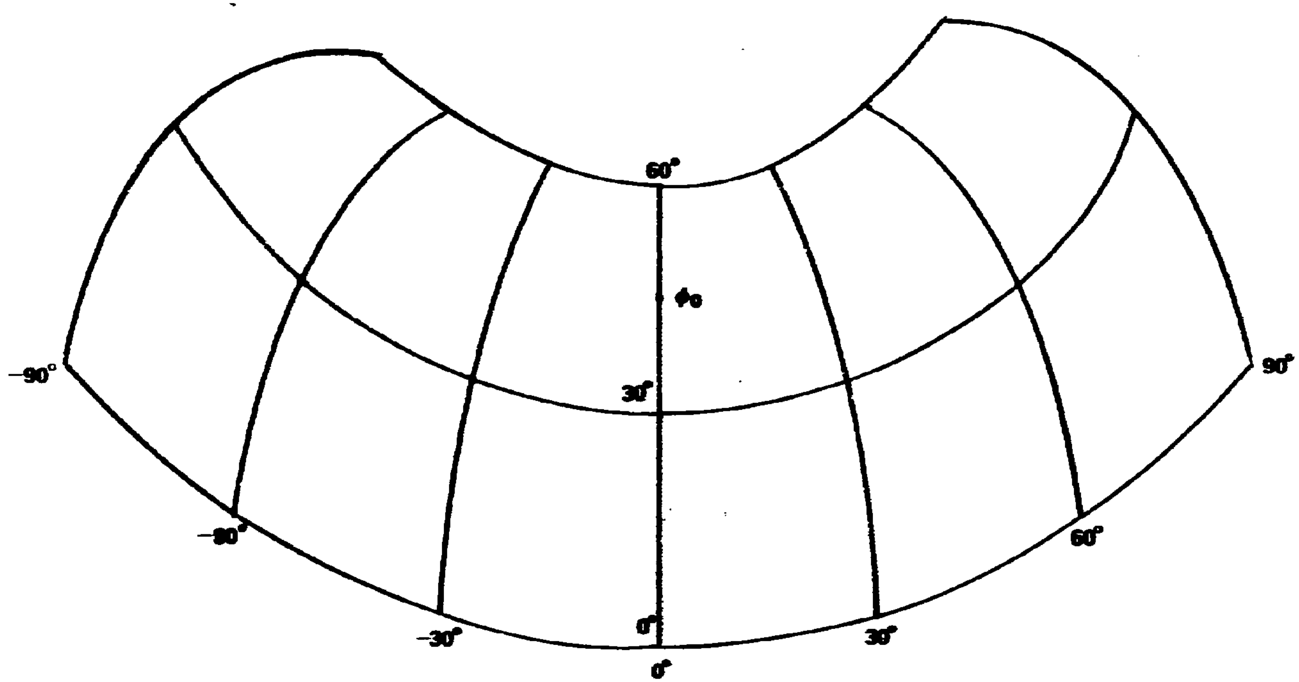


Figure 4.4.1. Bonne's equal area projection

4.5 Cylindrical Projection [2], [15], [20], [22]

In this projection, the meridians and parallels are straight lines, perpendicular to one another. The lines representing the meridians are equally spaced along the equator. It remains to space the parallels according to the requirement that an area on the projection is equal to the corresponding area on the authalic sphere.

Since the meridians are perpendicular to the equator, and equally spaced, the abscissa is

$$x = R \cdot S \cdot (\lambda - \lambda_0) \tag{4.5.1}$$

where S is the scale factor, and λ and λ_0 are in radians. The longitude of the central meridian is λ_0.

Consider the projection from the authalic sphere to a plane, rectangular coordinate system. For the plane

$$ds^2 = dx^2 + dy^2$$

$$\left.\begin{aligned} E &= 1 \\ G &= 1 \end{aligned}\right\} . \tag{4.5.2}$$

For the sphere

$$\begin{aligned} ds^2 &= R^2\, d\phi^2 + R^2 \cos^2 d\lambda^2 \\ e &= R^2 \\ g &= R^2 \cos^2 \phi . \end{aligned} \tag{4.5.3}$$

The condition for equivalency of area is, again

$$eg - f^2 = \begin{vmatrix} E & F \\ F & G \end{vmatrix} \begin{vmatrix} \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial \lambda} \\ \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial \lambda} \end{vmatrix}^2 . \tag{4.5.4}$$

Since both systems are orthogonal, $f = F = 0$. Substituting this into (4.5.4)

$$eg = EG \begin{vmatrix} \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial \lambda} \\ \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial \lambda} \end{vmatrix}^2 . \tag{4.5.5}$$

From (4.5.1), and omitting the scale factor,

$$\frac{\partial x}{\partial \phi} = 0 \tag{4.5.6}$$

$$\frac{\partial x}{\partial \lambda} = R . \tag{4.5.7}$$

Substituting (4.5.2), (4.5.3), (4.5.6), and (4.5.7) into (4.5.5)

$$R^4 \cos^2 \phi = \begin{vmatrix} \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \lambda} \\ 0 & R \end{vmatrix}^2$$

$$= R^2 \left(\frac{\partial y}{\partial \phi}\right)^2$$

$$\frac{\partial y}{\partial \phi} = R \cos \phi . \tag{4.5.9}$$

Integrating (4.5.9)

$$y = R \sin \phi + c . \tag{4.5.10}$$

The constant c in (4.5.10) can be zero by selecting the origin on the equator. Including the scale factor, (4.5.10) becomes

$$y = R \cdot S \sin \phi . \tag{4.5.11}$$

The same result can be obtained in a different manner. The area of the zone below latitude ϕ on the authalic sphere is

$$A = 2\pi R^2 \sin \phi . \tag{4.5.12}$$

The area on a cylinder tangent to this sphere at the equator is

$$A = 2\pi R y . \tag{4.5.13}$$

Equating (4.5.12) and (4.5.13)

$$2\pi R y = 2\pi R^2 \sin \phi$$

$$y = R \sin \phi .$$

This duplicates (4.5.11). The second method was the one originally employed, and gave the projection its name.

The grid resulting from (4.5.1) and (4.5.11) is given in Figure 4.5.1. Observe that distortion is intense at higher latitudes, and the projection can be of real service only near the equator. This consideration has limited the usefulness of the projection. Table 4.5.1 gives the plotting coordinates. This projection is also graphically constructed.

This projection can be made oblique by applying the rotation formulas of Chapter 2. If this is done, the area adjacent to the great circle tangent to the cylinder has a region fairly free of distortion.

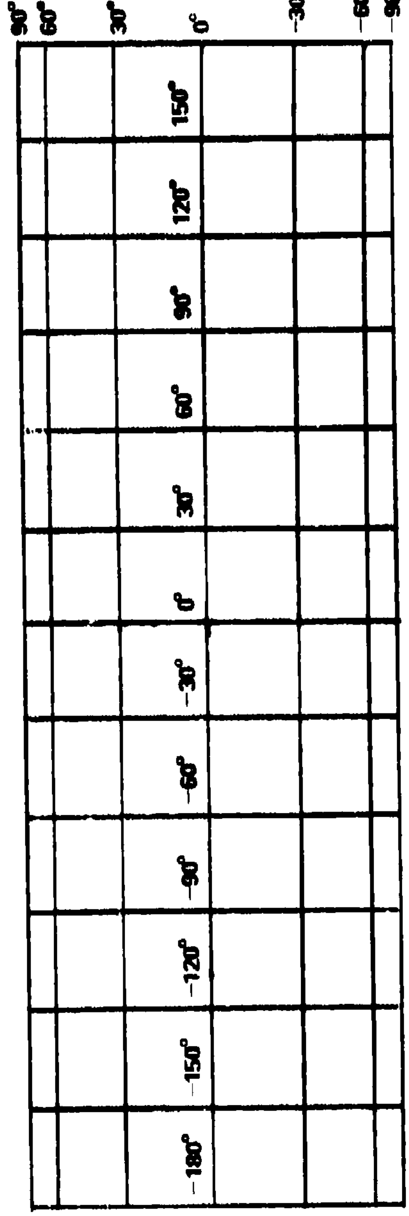


Figure 4.5.1. Cylindrical equal area projection

Table 4.5.1. Cylindrical Equal Area Projection.

Equal Area Cylindrical

Latitude*	Longitude*	X**	Y**
0.0000	0.0000	0.000	0.000
0.0000	30.0000	3.340	0.000
0.0000	60.0000	6.679	0.000
0.0000	90.0000	10.019	0.000
0.0000	120.0000	13.359	0.000
0.0000	150.0000	16.698	0.000
0.0000	180.0000	20.038	0.000
30.0000	0.0000	0.000	3.189
30.0000	30.0000	3.340	3.189
30.0000	60.0000	6.679	3.189
30.0000	90.0000	10.019	3.189
30.0000	120.0000	13.359	3.189
30.0000	150.0000	16.698	3.189
30.0000	180.0000	20.038	3.189
60.0000	0.0000	0.000	5.524
60.0000	30.0000	3.340	5.524
60.0000	60.0000	6.679	5.524
60.0000	90.0000	10.019	5.524
60.0000	120.0000	13.359	5.524
60.0000	150.0000	16.698	5.524
60.0000	180.0000	20.038	5.524
90.0000	0.0000	0.000	6.378
90.0000	30.0000	3.340	6.378
90.0000	60.0000	6.679	6.378
90.0000	90.0000	10.019	6.378
90.0000	120.0000	13.359	6.378
90.0000	150.0000	16.698	6.378
90.0000	180.0000	20.038	6.378

$\phi_0 = 0°$ *Degrees

$\lambda_0 = 0°$ **Meters

4.6 Sinusoidal Projection [8], [15], [20]

The Sinusoidal projection, also called the Sanson-Flamsteed projection, is a projection of the entire authalic sphere. Essentially, it is derived from the Bonne projection by setting $\phi_0 = 0$. Then, ρ approaches infinity.

On the authalic sphere,

$$ds^2 = R^2\, d\phi^2 + R^2 \cos^2\phi\, d\lambda^2$$

$$\left.\begin{aligned} e &= R^2 \\ g &= R^2 \cos^2\phi \end{aligned}\right\} . \tag{4.6.1}$$

On the plane, in Cartesian coordinates,

$$ds^2 = dx^2 + dy^2$$

$$\left.\begin{aligned} E &= 1 \\ G &= 1 \end{aligned}\right\} . \tag{4.6.2}$$

On the sphere,

$$f = 0 \tag{4.6.3}$$

and along the central meridian,

$$F = 0 . \tag{4.6.4}$$

The equal area condition is

$$eg - f^2 = \begin{vmatrix} E & F \\ F & G \end{vmatrix} \begin{vmatrix} \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial \lambda} \\ \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial \lambda} \end{vmatrix}^2 . \tag{4.6.5}$$

Along the meridian

$$y = R\phi . \tag{4.6.6}$$

Therefore,

$$\left.\begin{aligned} \frac{\partial y}{\partial \phi} &= R \\ \frac{\partial y}{\partial \phi} &= 0 \end{aligned}\right\} . \tag{4.6.7}$$

Substitute (4.6.1), (4.6.2), (4.6.3), (4.6.4), and (4.6.7) into (4.6.5).

$$R^4 \cos^2\phi = \begin{vmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \lambda} \\ R & 0 \end{vmatrix}^2$$

$$= R^2 \left(\frac{\partial x}{\partial \lambda}\right)^2$$

$$R \cos\phi = \frac{\partial x}{\partial \lambda} . \tag{4.6.8}$$

Convert (4.6.8) into an ordinary differential equation, and integrate.

$$dx = R \cos\phi \, d\lambda$$

$$x = \lambda R \cos\phi + c .$$

Since $x = 0$, when $\lambda = 0$, $c = 0$.

$$x = \lambda RS \cos\phi . \tag{4.6.9}$$

From (4.6.6)

$$y = R \cdot S \cdot \phi . \tag{4.6.10}$$

Figure 4.6.1 is a Sinusoidal projection of the earth. All of the parallels are straight lines. The meridians are sinusoidal curves. The central meridian and the equator are straight lines.

The Sinusoidal projection is used for geographical maps. The distortion at extreme latitudes and longitudes is simply ignored. A plotting table is in Table 4.6.1.

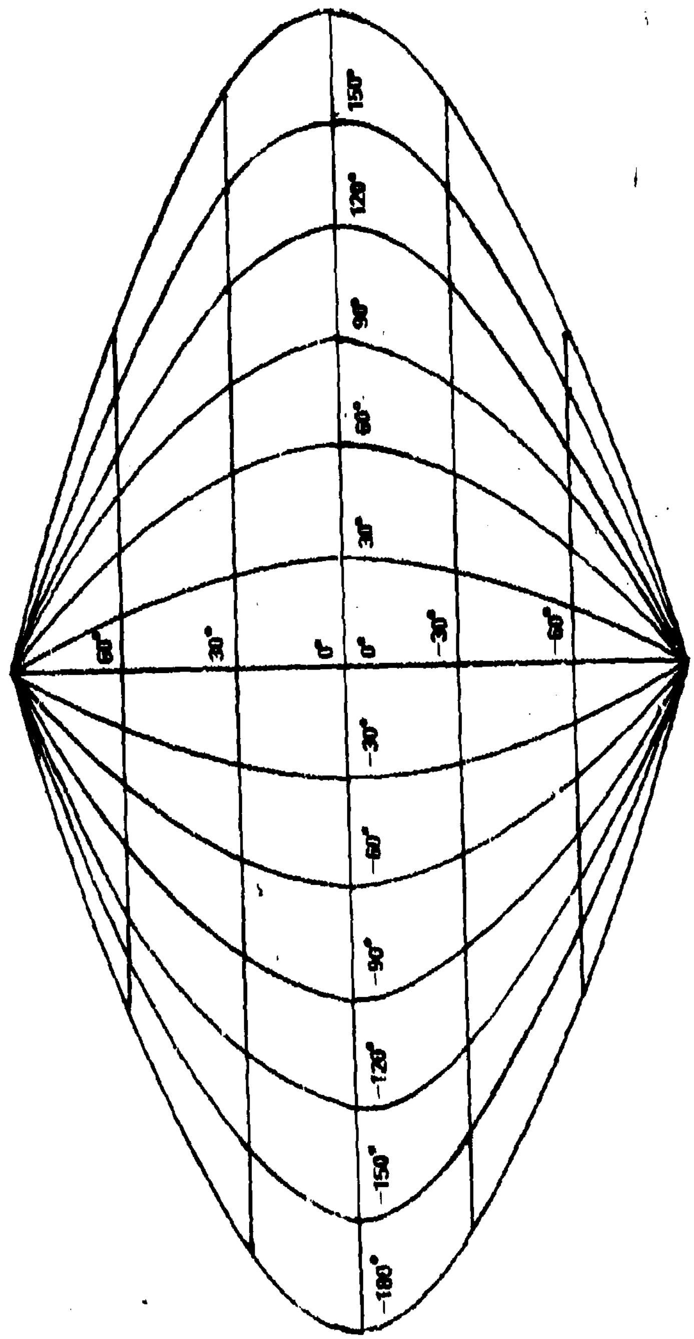


Figure 4.6.1. Sinusoidal projection

Table 4.6.1. Sinusoidal Projection.

Sinusoidal

Latitude*	Longitude*	X**	Y**
0.0000	0.0000	0.000	0.000
0.0000	30.0000	3.340	0.000
0.0000	60.0000	6.679	0.000
0.0000	90.0000	10.019	0.000
0.0000	120.0000	13.359	0.000
0.0000	150.0000	16.698	0.000
0.0000	180.0000	20.038	0.000
30.0000	0.0000	0.000	3.340
30.0000	30.0000	2.892	3.340
30.0000	60.0000	5.784	3.340
30.0000	90.0000	8.677	3.340
30.0000	120.0000	11.569	3.340
30.0000	150.0000	14.461	3.340
30.0000	180.0000	17.353	3.340
60.0000	0.0000	0.000	6.679
60.0000	30.0000	1.670	6.679
60.0000	60.0000	3.340	6.679
60.0000	90.0000	5.009	6.679
60.0000	120.0000	6.679	6.679
60.0000	150.0000	8.349	6.679
60.0000	180.0000	10.019	6.679
90.0000	0.0000	0.000	10.019
90.0000	30.0000	-.000	10.019
90.0000	60.0000	-.000	10.019
90.0000	90.0000	-.000	10.019
90.0000	120.0000	-.000	10.019
90.0000	150.0000	-.000	10.019
90.0000	180.0000	-.000	10.019

$\phi_0 = 0°$ *Degrees

$\lambda_0 = 0°$ **Meters

4.7 Mollweide Projection [15], [22]

The Mollweide (Elliptic) projection of the authalic sphere is derived from the construction in Figure 4.7.1. All of the meridians are ellipses. The central meridian is a retilinear ellipse, or straight line, and the 90° meridians are ellipses of eccentricity zero, or circular arcs. The equator and parallels are straight lines perpendicular to the central meridian. The central meridian and the equator are true length.

The main problem in this projection is spacing the parallels so that the property of equivalence of area is maintained. To do this, apply the law of equal surface from the authalic sphere to the planar map.

The area of the circle centered at O is

$$A_1 = \pi r^2 . \tag{4.7.1}$$

This is to be equal in area to a hemisphere

$$A_1 = 2\pi R^2 . \tag{4.7.2}$$

Equating (2.7.1) and (4.7.2)

$$\pi r^2 = 2\pi R^2$$

$$r^2 = 2R^2 \tag{4.7.3}$$

$$r = \sqrt{2}\, R . \tag{4.7.4}$$

Consider Figure 4.7.1. The area between latitude ϕ on the sphere, and the equator is

$$A = 2\pi R^2 \sin\phi . \tag{4.7.5}$$

This area is equal to the area AEFI on the figure. For a circle inscribed within an ellipse, where the radius of the circle is one half of the semi-major axis the area BDGH equals one half of the area AEFI [17]. Consider half of the area BDGH, that is area CDGO. This area is composed of the triangle OCD and the sector ODG. The area of the triangle is

$$A_A = \frac{1}{2} r \sin\theta \, r \cos\theta$$

$$= \frac{r^2}{4} \sin 2\theta . \tag{4.7.6}$$

The area of the sector is

$$A_S = \frac{r^2\theta}{2} . \tag{4.7.7}$$

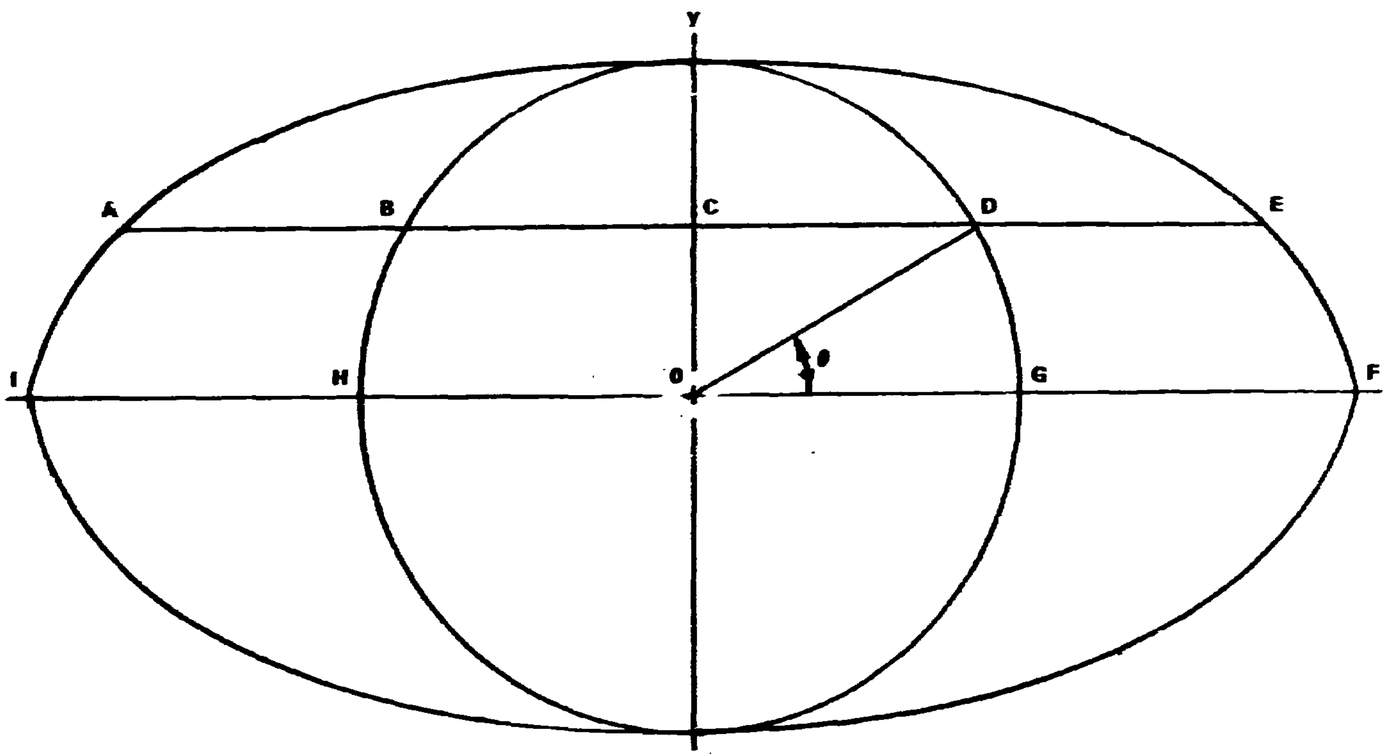


Figure 4.7.1. Geometry of the Mollweide projection

Equate the spherical area and the map area through (4.7.5), (4.7.6), and (4.7.7).

$$2\pi R^2 \sin\phi = 4\left(\frac{1}{2} r^2 \theta + \frac{1}{4} r^2 \sin 2\theta\right)$$

$$\pi R^2 \sin\phi = r^2 \theta + \frac{1}{2} r^2 \sin 2\theta . \tag{4.7.8}$$

Substitute (4.7.3) into (4.7.8).

$$\pi R^2 \sin\phi = 2R^2\theta + R^2 \sin 2\theta$$

$$\pi \sin\phi = 2\theta + \sin 2\theta . \tag{4.7.9}$$

We are now faced with a transcendental equation to be solved for θ. For limited accuracy, a graph of θ versus ϕ can be constructed, as in Figure 4.7.2, and values of θ read for given values of ϕ. However, for computer implementation of this projection, it is necessary to resort to a numerical solution.

Apply the Newton-Raphson method [14]. Write (4.7.9) as

$$f(\theta) = \pi \sin\phi - 2\theta - \sin 2\theta = 0 . \tag{4.7.10}$$

Differentiating (4.7.10)

$$f'(\theta) = -2 - 2\cos 2\theta . \tag{4.7.11}$$

The iterative solution of (4.7.9) for θ as a function of ϕ is

$$\theta_{n+1} = \theta_n - \frac{f(\theta_n)}{f'(\theta_n)} . \tag{4.7.12}$$

Substitute (4.7.10) and (4.7.11) into (4.7.12).

$$\theta_{n+1} = \theta_n + \frac{\pi \sin\phi - 2\theta_n - \sin 2\theta_n}{2 + 2\cos^2\theta_n}$$

where θ_n is in radians. This has a rapid convergence if the initial guess for θ is the given value of ϕ.

Once θ is found, the mapping equations quickly follow from Figure 4.7.1.

$$y = R \cdot S \sin\theta \tag{4.7.13}$$

$$x = \frac{(\lambda - \lambda_0)}{180} S \cdot 2r \cos\theta$$

$$= \frac{(\lambda - \lambda_0)}{90} RS \cos\theta \tag{4.7.14}$$

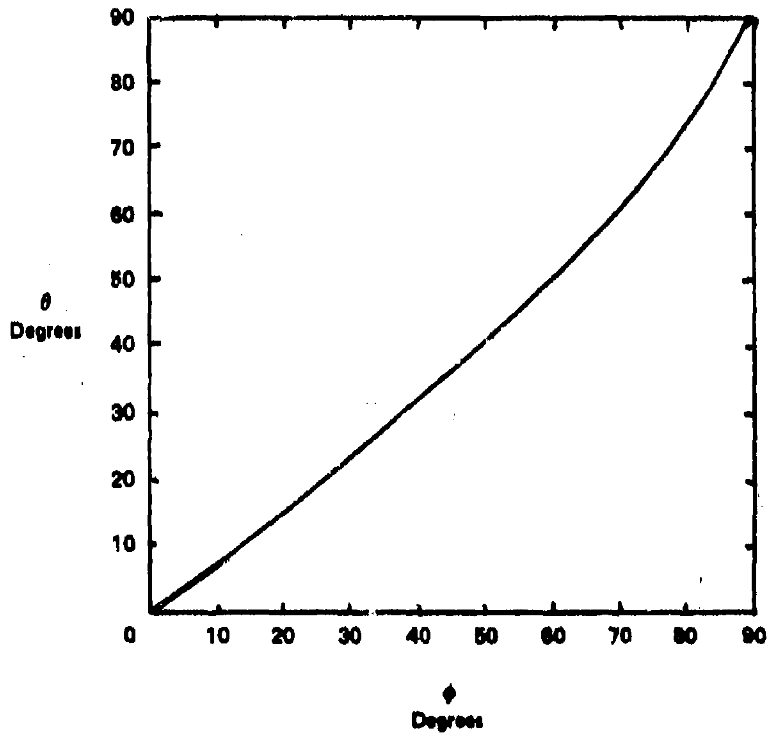


Figure 4.7.2. θ vs ϕ for the Mollweide projection

where r is given by (4.7.4), S is the scale factor, and λ and λ_0 is in radians. The central meridian has longitude λ_0.

The result of applying (4.7.13), (4.7.14), and the iteration is the grid of Figure 4.7.3. The distortion towards the poles is not as great as in the Sinusoidal projection, but it is more noticeable than in the Hammer-Aitoff projection. The chief use of the Mollweide projection is for geographical illustrations relating to area, where distortions are not disturbing.

Plotting tables for the Mollweide projection are given in Table 4.7.1. Latitudes 0° to 90°, in steps of 30°, and longitudes 0° to 180° in steps of 30°, are tabulated.

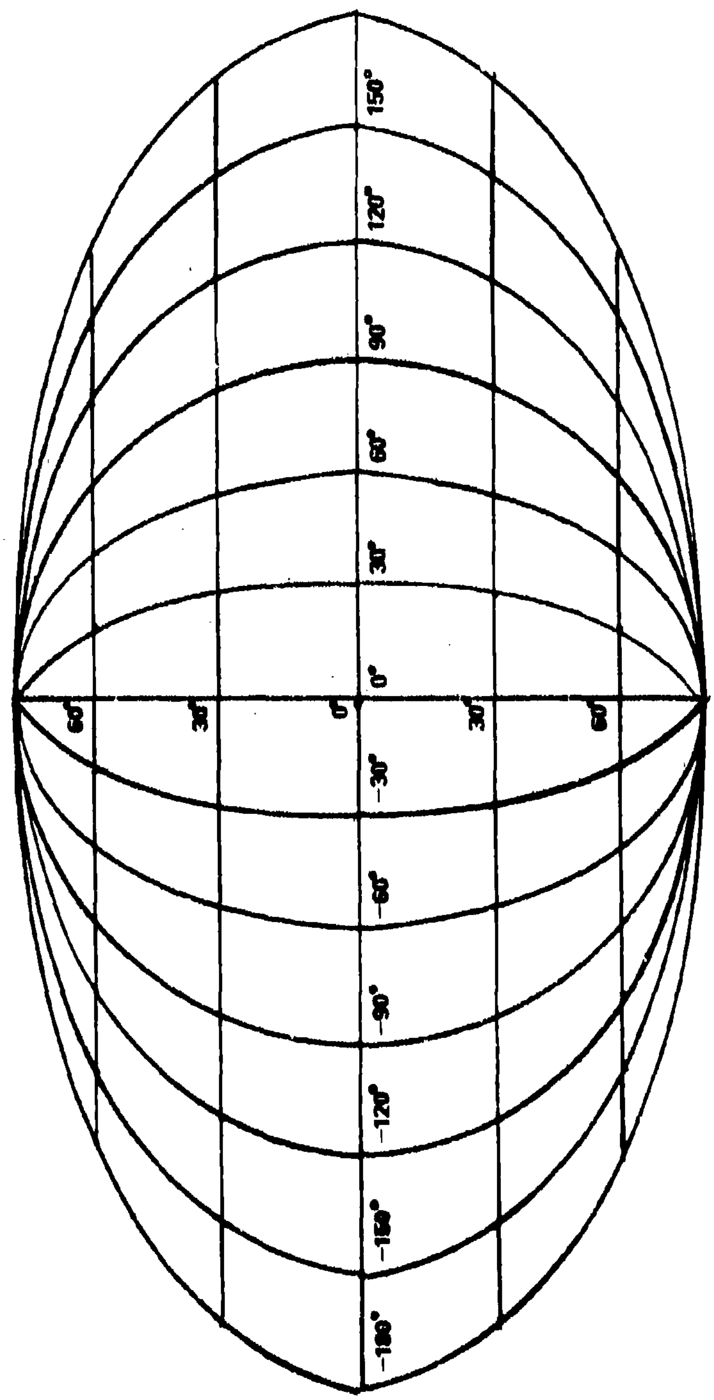


Figure 4.7.3. Mollweide projection

Table 4.7.1. Mollweide Projection.

Mollweide

Latitude*	Longitude*	X**	Y**
0.0000	0.0000	0.000	0.000
0.0000	30.0000	3.340	0.000
0.0000	60.0000	6.679	0.000
0.0000	90.0000	10.019	0.000
0.0000	120.0000	13.359	0.000
0.0000	150.0000	16.698	0.000
0.0000	180.0000	20.038	0.000
30.0000	0.0000	0.000	4.047
30.0000	30.0000	3.055	4.047
30.0000	60.0000	6.110	4.047
30.0000	90.0000	9.165	4.047
30.0000	120.0000	12.220	4.047
30.0000	150.0000	15.275	4.047
30.0000	180.0000	18.330	4.047
60.0000	0.0000	0.000	7.638
60.0000	30.0000	2.161	7.638
60.0000	60.0000	4.322	7.638
60.0000	90.0000	6.483	7.638
60.0000	120.0000	8.644	7.638
60.0000	150.0000	10.806	7.638
60.0000	180.0000	12.967	7.638
90.0000	0.0000	0.000	10.018
90.0000	30.0000	0.000	10.018
90.0000	60.0000	0.000	10.018
90.0000	90.0000	0.000	10.018
90.0000	120.0000	0.000	10.018
90.0000	150.0000	0.000	10.018
90.0000	180.0000	0.000	10.018

$\phi_0 = 0^\circ$
$\lambda_0 = 0^\circ$

*Degrees
**Meters

4.8 Parabolic Projection [8], [15], [22]

The Parabolic (Craster) projection of the entire world is shown in Figure 4.8.1. The parallels are straight lines parallel to a straight line equator. The meridians are parabolic arcs.

The Parabolic projection can be constructed from consideration of Figure 4.8.2. Let the equator be four units of length. Then, the central meridian is two units. A scale factor will be applied at the end of the derivation to convert the assumed units to the equatorial circumference. At present, R is the radius of the authalic sphere.

Consider the shaded area in Figure 4.8.2, bounded by an outer meridian, the central meridian, and the equator. The outer meridian is taken to be a parabola, $y^2 = x/2$, with its vertex at (0, 0). The mapping criterion requires that one quarter of the area on the authalic sphere will be equivalent to the shaded area between $x = 0$, and $x = 2$. Thus, one half of the zone between the equator, and a given parallel, ϕ, will be

$$A = \int_0^y (2 - x)\, dy$$

$$= \int_0^y (2 - 2y^2)\, dy$$

$$= 2y - \frac{2}{3} y^3 \Big|_0^y$$

$$= 2y - \frac{2}{3} y^3 . \tag{4.8.1}$$

The total area of the sphere is $4\pi R^2$. Substituting this value, and $y = 1$ into (4.8.1)

$$\pi R^2 = 4/3 \tag{4.8.2}$$

$$R = \sqrt{\frac{4}{3\pi}}$$

$$= 0.651470 .$$

Next, relate the map ordinate to authalic latitude. The area of a zone on the authalic sphere from the equator to latitude ϕ is $2\pi R^2 \sin\phi$. Half of this zone is then

$$A = \pi R^2 \sin\phi . \tag{4.8.3}$$

Equate (4.8.1) and (4.8.3)

$$2y - \frac{2}{3} y^3 = \pi R^2 \sin\phi . \tag{4.8.4}$$

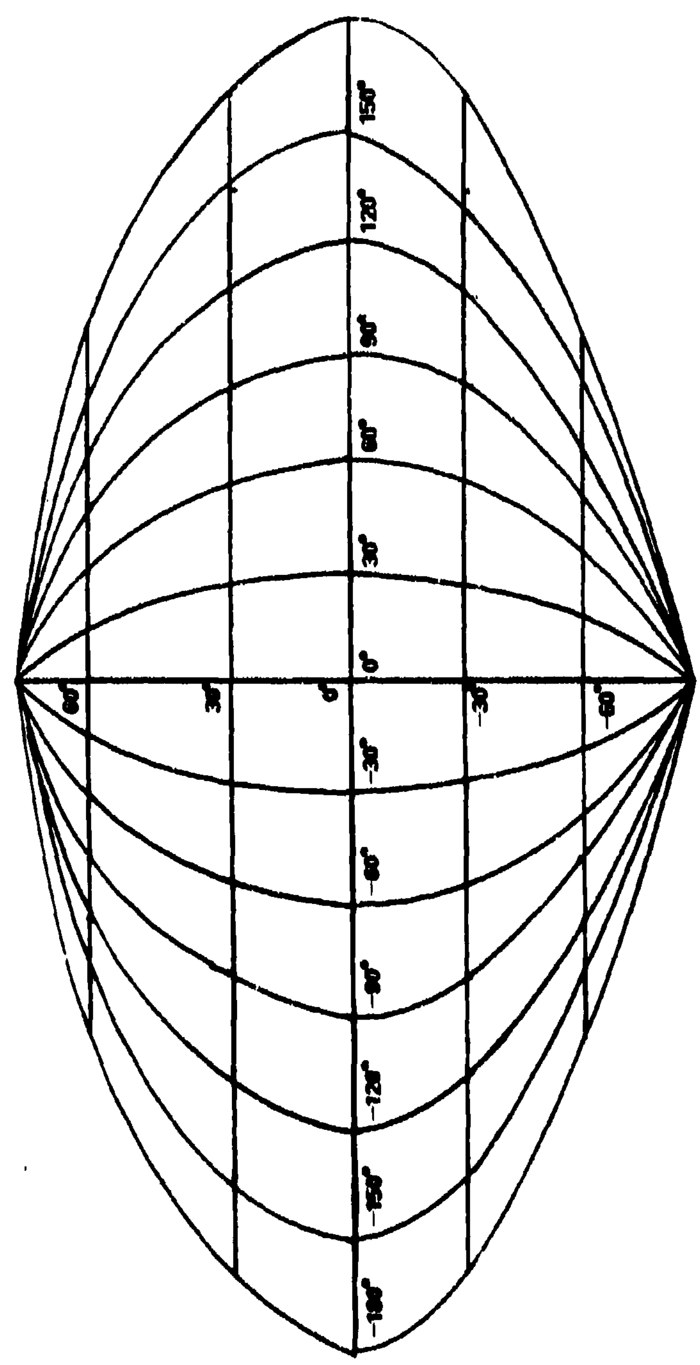


Figure 4.8.1. Parabolic projection

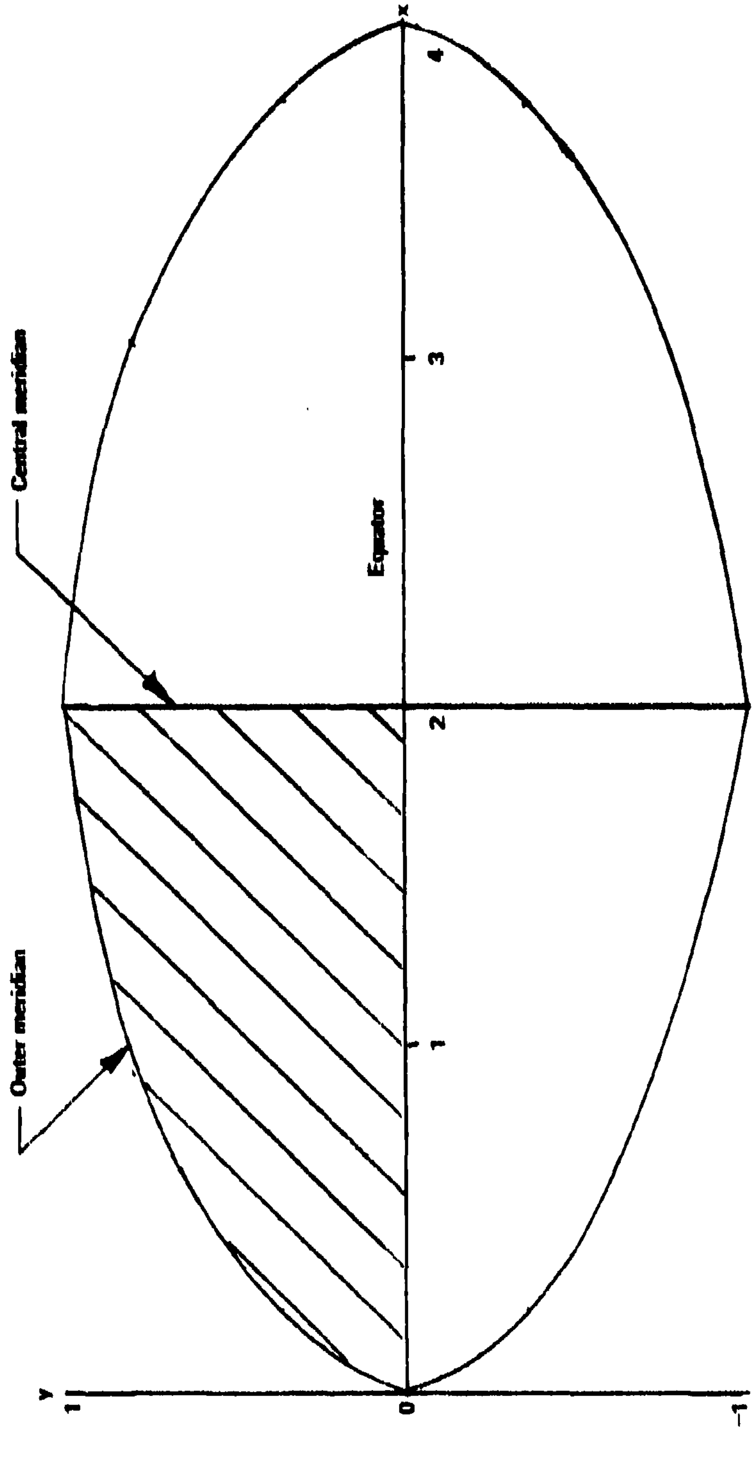


Figure 4.8.2 Geometry for the Parabolic projection

Substitute (4.8.2) into (4.8.4).

$$2y - \frac{2}{3} y^3 = \frac{4}{3} \sin \phi$$

$$y^3 - 3y + 2 \sin \phi = 0 .$$

A solution of this transcendental equation is

$$y = 2 \sin \phi/3 \qquad (4.8.5)$$

which can be verified by substitution. A scale factor, S, and radius, R, may be introduced into (4.8.5) to obtain the ordinate.

$$y = 2S \sin \phi/3 \cdot R . \qquad (4.8.6)$$

The abscissa may be obtained by the following development. The length of a parallel between the central meridian and the outer meridian is given by

$$\ell = 2 - 2y^2 . \qquad (4.8.7)$$

Substitute (4.8.5) into (4.8.7).

$$\begin{aligned} \ell &= 2(1 - 4 \sin^2 \phi/3) \\ &= 2\left(1 + 2 \cos \frac{2\phi}{3} - 2\right) \\ &= 2\left(2 \cos \frac{2\phi}{3} - 1\right). \qquad (4.8.8) \end{aligned}$$

The parallels are divided proportionally for the intersections of the meridians. From (4.8.8), and including the scale factor, S, and radius R,

$$\begin{aligned} x &= \frac{(\lambda - \lambda_0)}{180} \ell S \cdot R \\ &= \frac{(\lambda - \lambda_0)}{90} SR \left(2 \cos \frac{2\phi}{3} - 1\right). \qquad (4.8.9) \end{aligned}$$

In (4.8.9), $\lambda - \lambda_0$ is the difference in longitude between the given meridian and the central meridian, in degrees.

Since this is an equal area projection, its use is for statistical representation. No attempt is made to avoid distortion in angles and shapes. However, the distortion is less than in the Mollweide projection because the meridians and parallels do not intersect at such acute angles. Also, the symmetry and parabolic curves lend a certain aesthetic quality.

Equations (4.8.6) and (4.8.9) have computed in Table 4.8.1. This table gives the longitude from 0° to 180° in steps of 30°, and latitude from 0° to 90° in steps of 30°.

Table 4.8.1. Parabolic Projection.

Parabolic

Latitude*	Longitude*	X**	Y**
0.0000	0.0000	0.000	0.000
0.0000	30.0000	3.340	0.000
0.0000	60.0000	6.679	0.000
0.0000	90.0000	10.019	0.000
0.0000	120.0000	13.359	0.000
0.0000	150.0000	16.698	0.000
0.0000	180.0000	20.038	0.000
30.0000	0.0000	0.000	3.480
30.0000	30.0000	2.937	3.480
30.0000	60.0000	5.874	3.480
30.0000	90.0000	8.810	3.480
30.0000	120.0000	11.747	3.480
30.0000	150.0000	14.684	3.480
30.0000	180.0000	17.621	3.480
60.0000	0.0000	0.000	5.853
60.0000	30.0000	1.777	5.853
60.0000	60.0000	3.554	5.853
60.0000	90.0000	5.331	5.853
60.0000	120.0000	7.108	5.853
60.0000	150.0000	8.885	5.853
60.0000	180.0000	10.662	5.853
90.0000	0.0000	0.000	10.019
90.0000	30.0000	-.000	10.019
90.0000	60.0000	-.000	10.019
90.0000	90.0000	-.000	10.019
90.0000	120.0000	-.000	10.019
90.0000	150.0000	-.000	10.019
90.0000	180.0000	-.001	10.019

$\phi_0 = 0°$ *Degrees
$\lambda_0 = 0°$ **Meters

4.9 Hammer-Aitoff Projection

The Hammer-Aitoff projection, shown in Figure 4.9.1, is derived by a mathematical manipulation of the Azimuthal Equal Area projection of Section 4.3. In the Hammer-Aitoff projection, the sphere is represented within an ellipse, with semi-major axis twice the length of the semi-minor axis. In this respect, it is similar to the Mollweide projection. However, in the Hammer-Aitoff projection, the parallels are curved lines, rather than straight.

The grid of meridians and parallels is obtained by the orthogonal projection of the Azimuthal Equal Area projection, Equatorial Case, onto planes making angles of 60° to the plane of the Azimuthal projection.

Figure 4.9.2 demonstrates the means of projection. In this figure, we are looking upon the edges of the planes, which appear as straight lines. Since the angle between the planes is 60°, DO = 2AO, and OB = 2OC. Thus, the total length DO plus OC is entire equator, as AB is half of the equator. It is assumed that for the Hammer-Aitoff projection the total map of the authalic sphere is obtained by unfolding DO and OC into a plane DOC, with O as the position of the central meridian.

In this projection, the ordinate is not modified from a comparable point on the Azimuthal Equal Area projection.

Converting to the coordinates of the auxiliary system (4.3.3) and (4.3.4) become

$$x = R \cdot S \cdot \sqrt{2(1 - \sin h)}\ \sin\alpha \tag{4.9.1}$$

$$y = R \cdot S \cdot \sqrt{2(1 - \sin h)}\ \cos\alpha\,. \tag{4.9.2}$$

From (2.10.4) and (2.10.5), for $\phi_0 = 0°$

$$\sin h = \cos\phi \cos\lambda_a \tag{4.9.3}$$

$$\tan\alpha = \frac{\sin\lambda_a}{\tan\phi}$$

$$\alpha = \tan^{-1}\left(\frac{\sin\lambda_a}{\tan\phi}\right) \tag{4.9.4}$$

where λ_a is the latitude on the azimuthal projection.

Substitute (4.9.3) and (4.9.4) into (4.9.1) and (4.9.2), and let $\lambda_a = \lambda/2$.

$$x = 2 \cdot R \cdot S \sqrt{2(1 - \cos\phi \cos\lambda/2)}\ \sin\left[\tan^{-1}\left(\frac{\sin\lambda/2}{\tan\phi}\right)\right] \tag{4.9.5}$$

Continued

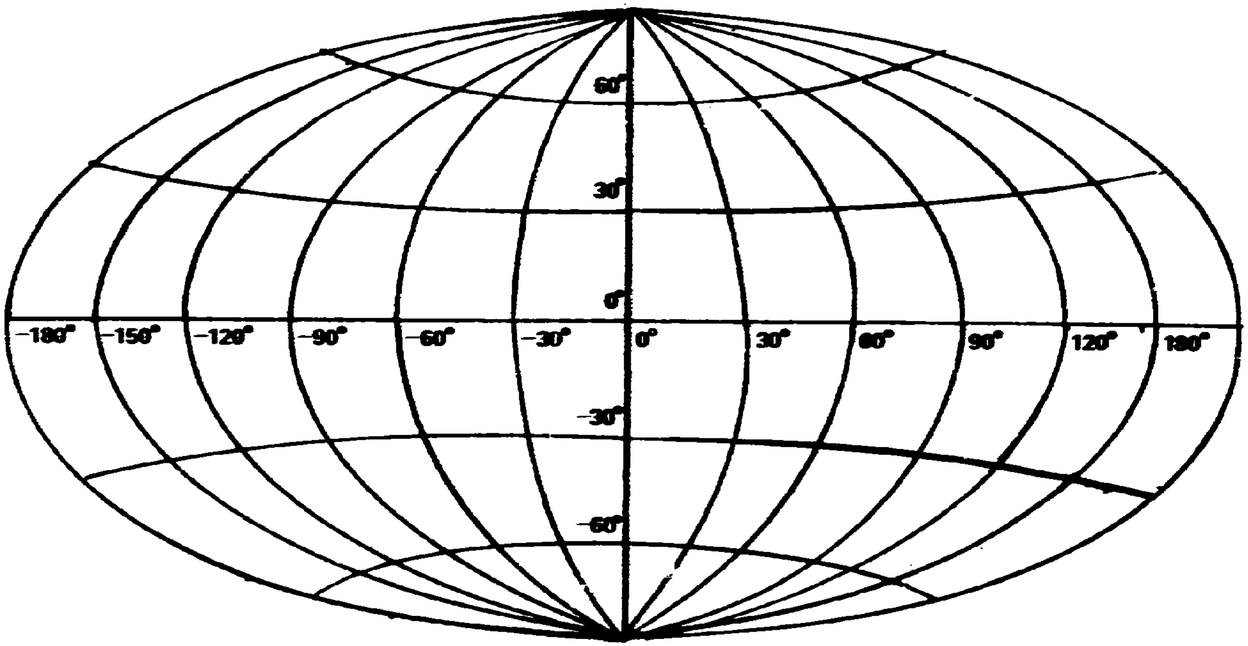


Figure 4.9.1. Hammer-Aitoff projection

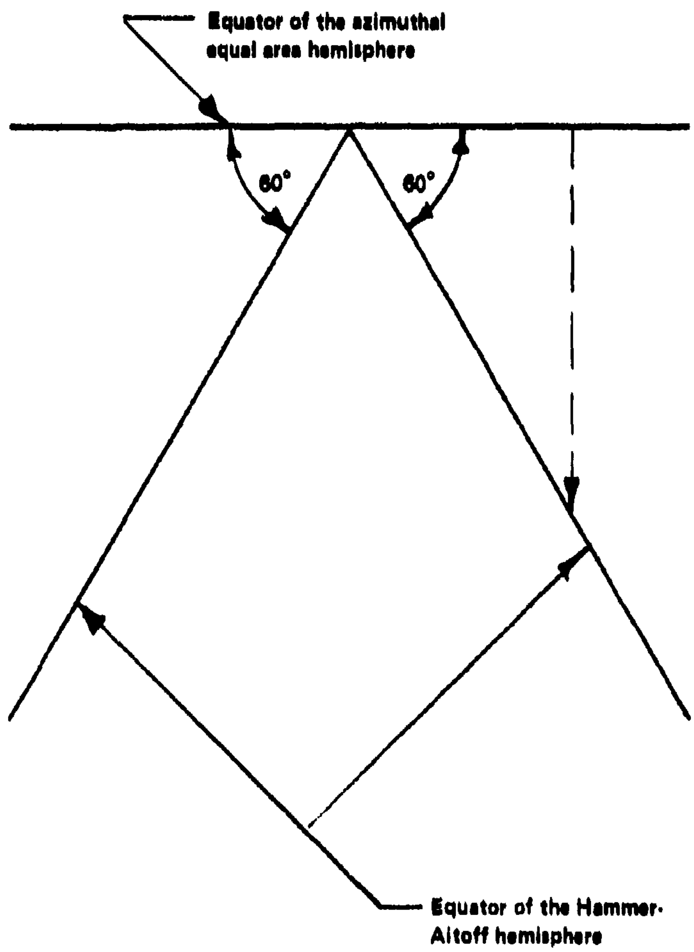


Figure 4.9.2. Geometry of the Hammer-Aitoff projection

$$y = R \cdot S \sqrt{2(1 - \cos\phi \cos\lambda/2)} \cos\left[\tan^{-1}\left(\frac{\sin\lambda/2}{\tan\phi}\right)\right]. \qquad (4.9.5)$$

Equations (4.9.5) gave the plotting relationship of Table 4.9.1.

That the area enclosed by the ellipse of the Hammer-Aitoff projection, corresponding to the entire sphere, is twice the area of the Azimuthal projection, corresponding to a hemisphere, follows easily from the geometry of the ellipse with a circle of the radius of the semi-minor axis inscribed [17].

In Figure 4.9.1, the central meridian and the equator are the only straight lines in the grid. The rest of the meridians and parallels are curves. The curvature of the parallels with respect to the meridians is such that there is less angular distortion than appears at higher latitudes, and more distant longitudes in the Mollweide projection.

The Hammer-Aitoff projection is used primarily for statistical representation of data. The distortion that occurs is overlooked.

Table 4.8.1. Hammer-Aitoff Projection.

Hammer-Aitoff Projection

Latitude*	Longitude*	X**	Y**
0.0000	0.0000	.000	.000
0.0000	30.0000	3.330	.000
0.0000	60.0000	6.603	.000
0.0000	90.0000	9.763	.000
0.0000	120.0000	12.756	.000
0.0000	150.0000	15.531	.000
0.0000	180.0000	18.040	.000
30.0000	0.0000	0.000	3.302
30.0000	30.0000	2.984	3.328
30.0000	60.0000	5.905	3.409
30.0000	90.0000	8.700	3.552
30.0000	120.0000	11.303	3.768
30.0000	150.0000	13.640	4.076
30.0000	180.0000	15.623	4.510
60.0000	0.0000	0.000	6.378
60.0000	30.0000	1.917	6.415
60.0000	60.0000	3.767	6.526
60.0000	90.0000	5.482	6.714
60.0000	120.0000	6.987	6.987
60.0000	150.0000	8.198	7.351
60.0000	180.0000	9.020	7.812
90.0000	0.0000	-.000	9.020
90.0000	30.0000	.000	9.020
90.0000	60.0000	.000	9.020
90.0000	90.0000	.000	9.020
90.0000	120.0000	.000	9.020
90.0000	150.0000	.000	9.020
90.0000	180.0000	.000	9.020

$\phi_0 = 0^\circ$ *Degrees
$\lambda_0 = 0^\circ$ **Meters

4.10 Interrupted Projections [22]

Interrupted projections of the authalic sphere are a means of reducing maximum distortion at the expense of continuity of the map. Figures 4.10.1, 4.10.2, and 4.10.3 show interrupted Sinusoidal Mollweide, and Parabolic projections, respectively.

Certain meridians are chosen as reference meridians, which are straight lines. The equator is also a straight line. Then other meridians are chosen where the breaks will occur. Note that it is not necessary for reference meridian to appear in both hemispheres. One can choose a half meridian in either hemisphere.

The parallels are spaced in the same manner as in the regular Sinusoidal, Mollweide or Parabolic projections. The difference comes in the method of handling the spacing of the meridians. Each reference meridian becomes the axis of the coordinate system, and the abscissa is marked, east or west, until a break is reached. Then, one goes to the next reference meridian, and repeats the process.

This procedure has been applied to the Sinusoidal, Mollweide, and Parabolic projections. In each case, the respective plotting equations of sections 4.6, 4.7, and 4.8 have been used.

The grids which result from this method are rather exotic in appearance. Distortion, since it is greatest at the farthest longitude from the central meridian, is always significantly decreased. These maps are generally used as statistical representations, so the breaks cannot create undue hardships. The breaks are chosen to appear in regions of little interest in order to better represent regions of greater interest.

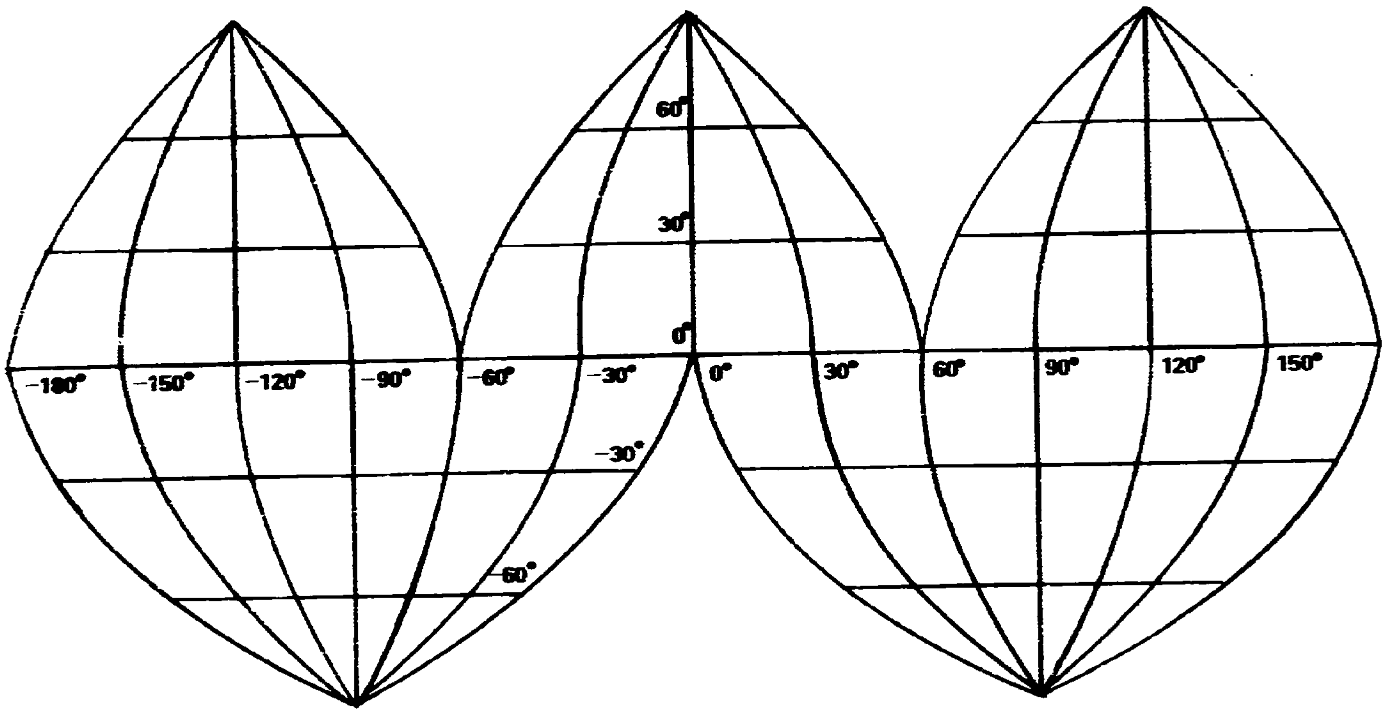


Figure 4.10.1. Interrupted Sinusoidal projection

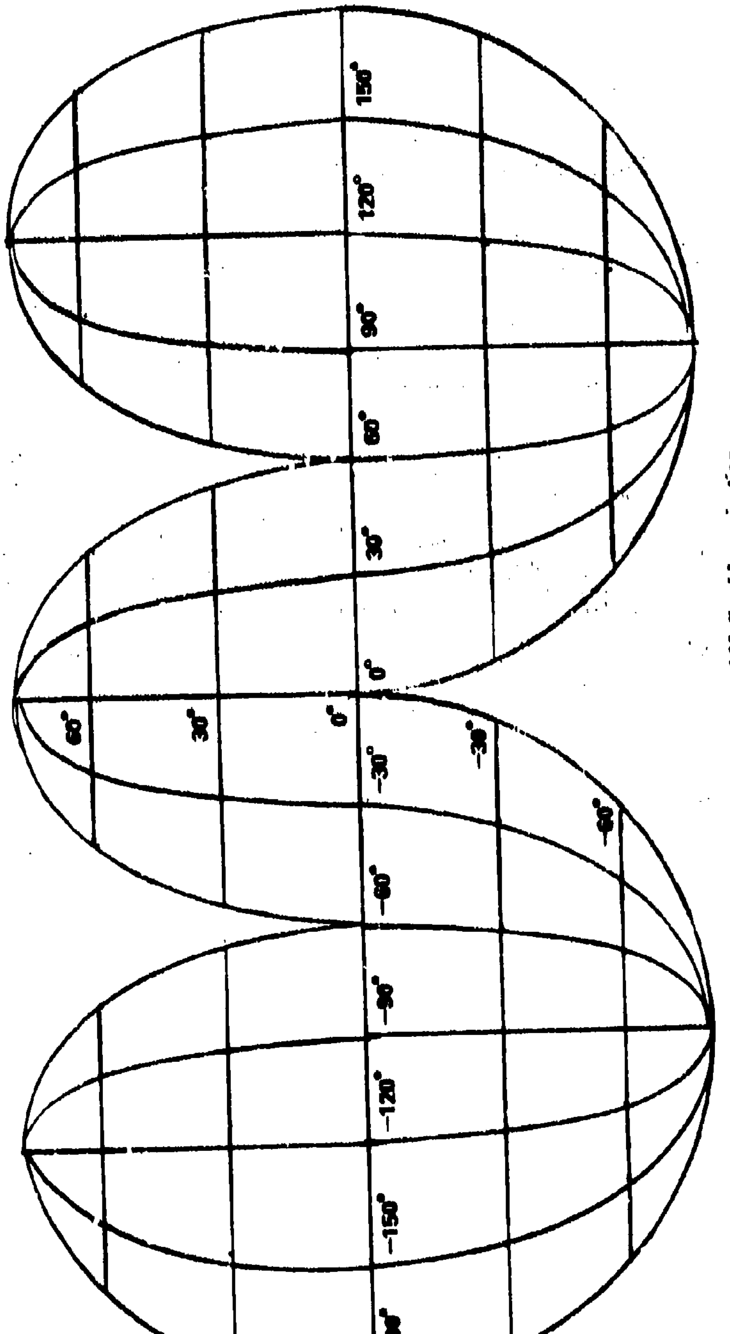


Figure 4.10.2 Interrupted Mollweide projection

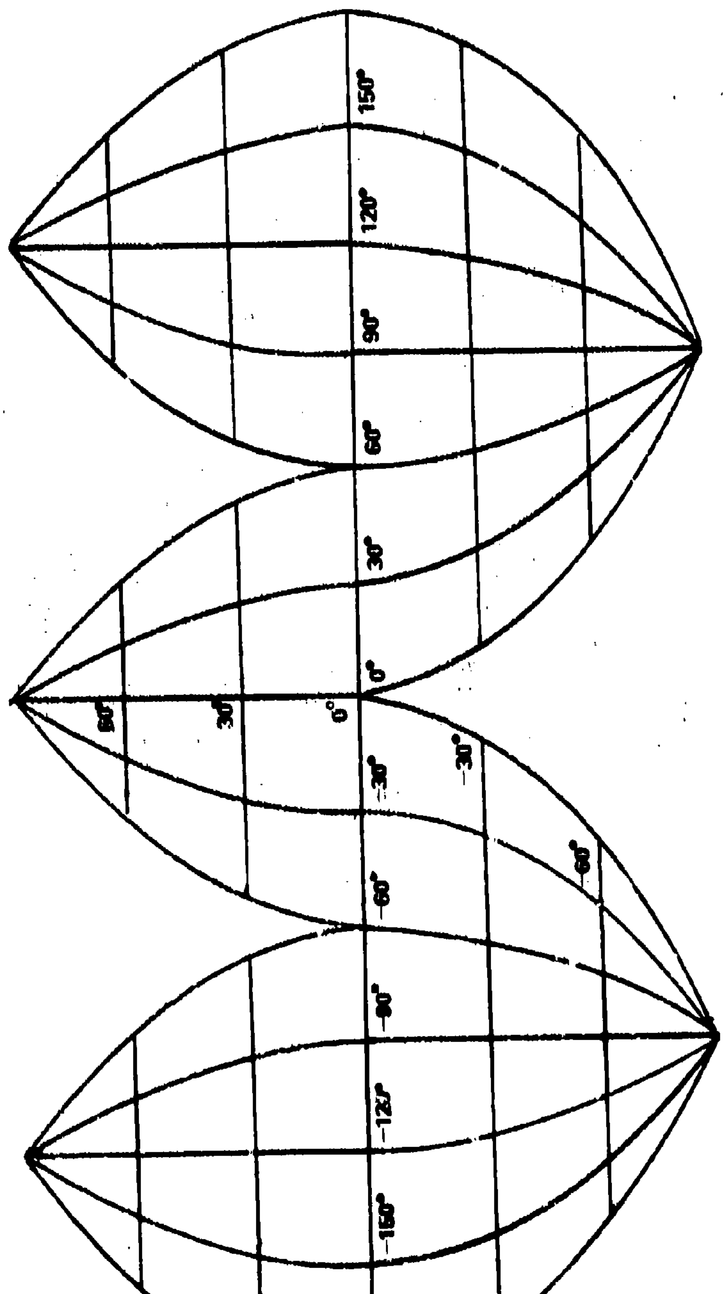


Figure 4.10.3. Interrupted Parabolic projection

4.11 Werner's Projection [2], [20]

Werner's projection is obtained from Bonne's projection by letting $\phi_0 = 90°$. From (4.4.5), we have

$$\rho = -\int_{\pi/2}^{\phi} R \, d\phi$$

$$= R(\pi/2 - \phi) . \qquad (4.11.1)$$

From (4.4.8), after substituting (4.11.1)

$$\theta = \frac{\lambda R \cos\phi}{R(\pi/2 - \phi)}$$

$$= \frac{\lambda \cos\phi}{\pi/2 - \phi} . \qquad (4.11.2)$$

If the origin is chosen at the pole, the Cartesian plotting equations become

$$\left.\begin{aligned} x &= R_s(\pi/2 - \phi) \sin\left(\frac{\lambda \cos\phi}{\pi/2 - \phi}\right) \\ y &= -R_s(\pi/2 - \phi) \cos\left(\frac{\lambda \cos\phi}{\pi/2 - \phi}\right) \end{aligned}\right\} . \qquad (4.11.3)$$

The grid corresponding to (4.11.3) is in Figure 4.11.1. The only straight line in this cardioid shaped projection is the central meridian. Note that distortion becomes excessive at the south pole, and at increased longitude from the central meridian. The parallels are still concentric circles. S is the scalefactor.

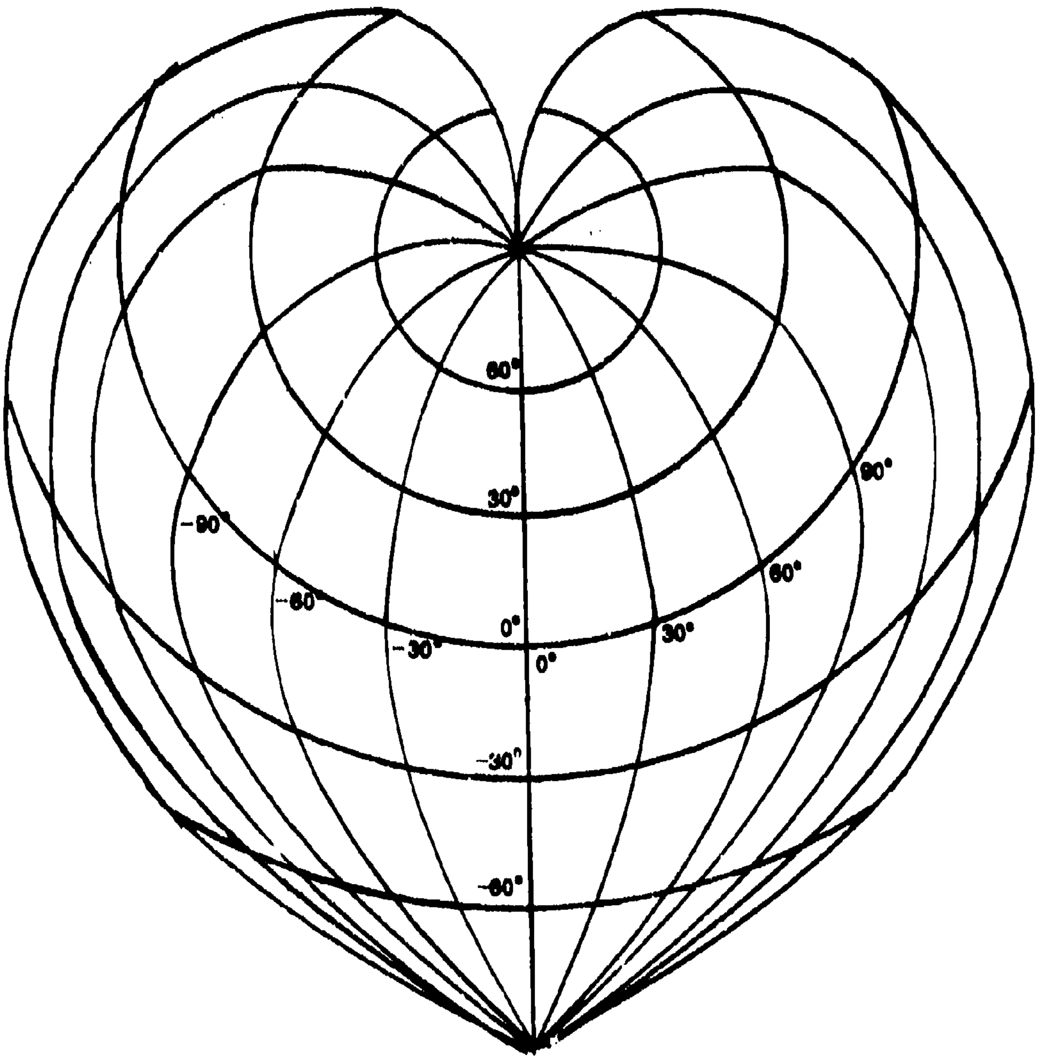


Figure 4.11.1. Werner's projection

4.12.1 Eumorphic Projection [22]

The Eumorphic projection in Figure 4.12.1 is essentially an arithmetic mean between the sinusoidal and Mollweide projectives. This will be the first projection in which one will be unable to obtain general x- and y-plotting coordinates.

The projection is obtained for each longitude by summing the distance along the central meridian to a given parallel for the sinusoidal and Mollweide, and then dividing by 2. The length of the parallel is then needed. This is obtained by requiring that the area between the equator, the central meridian, the given meridian, and the parallel under consideration be the same on the Eumorphic as on the Mollweide or the sinusoidal projection. This means obtaining comparable areas on the Mollweide or the sinusoidal by either a planimeter or integrating a polynomial which approximates the meridian curve. It is then necessary to obtain the x-coordinates on the Eumorphic by trial and error with a planimeter, or fitting a polynomial curve. This is time consuming, and has not been done in this report. However, Table 4.12.1 gives the y coordinates for the Eumorphic projection.

Table 4.12.1. Parallel Spacings on the Eumorphic Projection.

ϕ (Degrees)	Y (Meters)
0	0.000
30	0.547
60	1.061
90	1.490

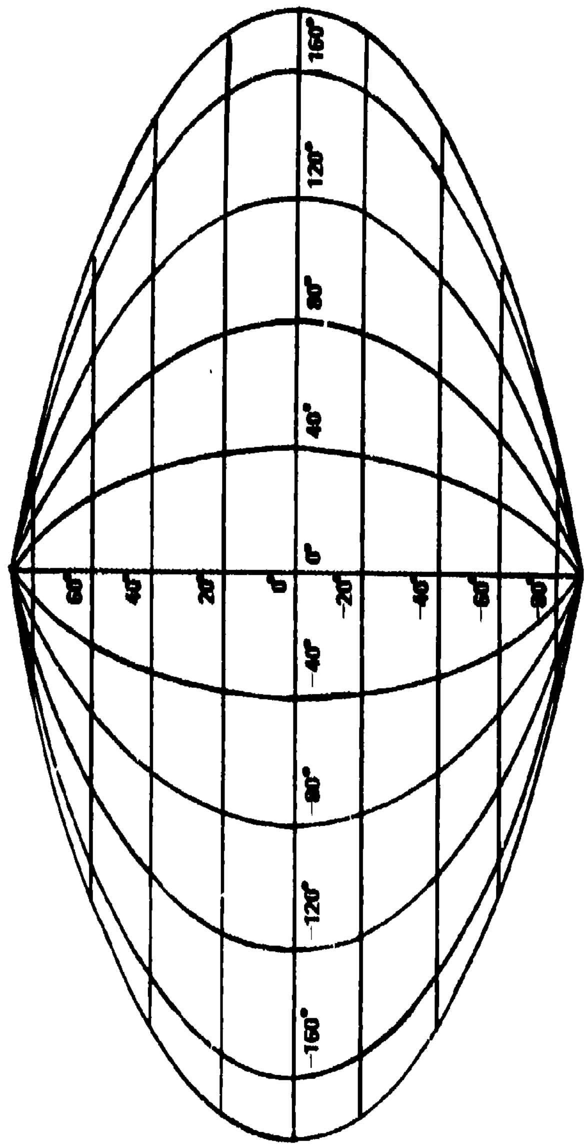


Figure 4.12.1. Eumorphic projection

4.13 Eckert's Projection [22], [24]

Eckert produced a total of six projections. The one that has received some fame is Eckert 4. In this projection, the central meridian is half the length of the equator. From Figure 4.13.1, the equator, parallels, and central meridian are straight lines. The method of choosing the parallels requires that the other meridians be elliptical curves.

In the projection, the spacing of the parallels decreases with latitude in a manner that makes this an equal area projection. The derivation is similar to that of the Mollweide.

The area of a hemisphere is

$$A_1 = 2\pi R^2 . \tag{4.13.1}$$

The area north of the equator on the Eckert projection is, from the Figure 4.13.1,

$$A_1 = 2r^2 + \frac{\pi r^2}{2} . \tag{4.13.2}$$

Equating (4.13.1) and (4.13.2)

$$\begin{aligned} 2\pi R^2 &= 2r^2 + \frac{\pi r^2}{2} \\ &= r^2\left(2 + \frac{\pi}{2}\right). \end{aligned} \tag{4.13.3}$$

The area on the hemisphere below latitude ϕ is

$$A_2 = 2\pi R^2 \sin\phi . \tag{4.13.4}$$

The corresponding area on the projection, including a rectangle two sectors, and two triangles, is

$$\begin{aligned} A_2 &= 2r \cdot r \sin\theta + \frac{2}{2} r \sin\theta \cdot r\cos\theta + \frac{2}{2} r^2\theta \\ &= r^2(2\sin\theta + \sin\theta\cos\theta + \theta) . \end{aligned} \tag{4.13.5}$$

Equating (4.13.4) and (4.13.5)

$$2\pi R^2 \sin\phi = r^2(2\sin\theta + \sin\theta\cos\theta + \theta) . \tag{4.13.6}$$

Equate (4.13.3) and (4.13.6) to obtain a relation between ϕ and θ.

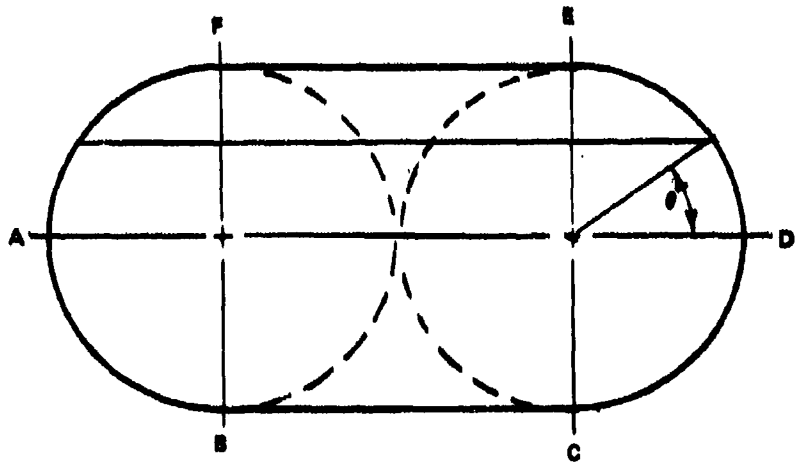


Figure 4.13.1. Geometry for Eckert's projection

$$r^2\left(2+\frac{\pi}{2}\right)\sin\phi = r^2(2\sin\theta+\sin\theta\cos\theta+\theta)$$

$$\left(2+\frac{\pi}{2}\right)\sin\phi = 2\sin\theta + \sin\theta\cos\theta + \theta\,. \qquad (4.13.7)$$

Again, we have an ugly transcendental equation to be solved for θ as a function of ϕ. And, again, the Newton-Raphson method will produce a result [14]. Armed with this, the Cartesian plotting coordinate are

$$\begin{aligned} x &= S\cdot R(1+\cos\theta)(\lambda-\lambda_0) \\ y &= S\cdot R\cdot\frac{\pi}{2}\sin\theta \end{aligned} \qquad (4.13.8)$$

Chapter 5

CONFORMAL PROJECTIONS

Conformal projections are those projections which locally maintain the shape of an area on the earth during the transformation to the mapping surface. One important aspect of this is that the orthogonal system of parallels and meridians on the spheroid appear as an orthogonal system on the map.

The process of conformal transformation may be accomplished in two ways. One of the ways is to transform from the spheroid onto a fictitious conformal sphere, and then apply the simpler spherical formulas to transform from the conformal sphere to the plane, cone, or cylinder. The second way is to accomplish a brute force transformation from the spheroid directly to the mapping surface. Both of these approaches will be considered. The results from either approach will be similar.

The conformal projections to be considered are the Mercator, the Lambert conformal, and the stereographic. Three variations of the Mercator will be discussed: the regular, the oblique, and the transverse. Lambert conformal projections will be represented by one, and two standard parallel cases. The polar, equatorial, and oblique versions of the stereographic will be derived.

All of the conformal projections will be characterized by the conformal relation introduced in Section 2.6. The fundamental quantities of the figure of the earth and the mapping surface will be related by

$$\frac{E}{e} = \frac{F}{f} = \frac{G}{g}.$$

In this equation, the capital letters refer to the mapping surface, and the small letters, to the chosen figure of the earth. Since we will be dealing with orthogonal systems, $f = F = 0$.

5.1 The Conformal Sphere [20], [23], [24]

The process of producing a conformal mapping of the earth onto a developable surface is aided by the fact that a succession of conformal transformations yields a conformal image of the original area on the final surface. It will be shown that it is possible to project the spheroid conformally onto a sphere of radius $R_c = \sqrt{R_p R_m}$, and maintain the quality of conformality for the subsequent transformation to the developable surface.

If one goes through the expansions of this double transformation, and then goes through the expansions for the direct transformation, he will observe that the results are similar, but not exact. The two approaches differ in higher order terms. Since these higher order terms contain powers of the eccentricity, e, the numerical difference is negligible. The process of expansion will not be attempted in this volume. The equations in this section will be derived in a form convenient for evaluation on a computer, and will be incorporated in the general mapping computer program of Appendix A.1.

Once the transformation from the spheroidal earth to the conformal sphere is complete, the formulas of trigonometry can be applied to transform from the conformal sphere to the mapping surface.

From (2.3.15), the first fundamental form of the spheroid is

$$(ds)^2 = R_m^2 (d\phi)^2 + R_p^2 \cos^2 \phi \, (d\lambda)^2$$

with fundamental quantities

$$\begin{aligned} e &= R_m^2 \\ g &= R_p^2 \cos^2 \phi \, . \end{aligned} \qquad (5.1.1)$$

The first fundamental form of the sphere is, from (2.3.14)

$$(ds)^2 = R_c^2 (d\Phi)^2 + R_c^2 \cos^2 \Phi \, (d\Lambda)^2$$

with fundamental quantities

$$\left.\begin{aligned} E' &= R_c^2 \\ G' &= R_c^2 \cos^2 \Phi \end{aligned}\right\} . \qquad (5.1.2)$$

For this conformal sphere, the conformal latitude and longitude are defined to be Φ, and Λ, respectively. The radius of the conformal sphere is R_c.

The conditions are applied that the conformal spherical latitude is a function of spheroidal geodetic latitude only, and conformal longitude is a linear function of spheroidal longitude. Mathematically, this is stated as

$$\left.\begin{aligned} \Phi &= \Phi(\phi) \\ \Lambda &= c\lambda + c_1 \end{aligned}\right\} . \qquad (5.1.3)$$

Applying the fundamental transformation matrix (2.7.11) to (5.1.2) and (5.1.3)

$$\left.\begin{aligned} E &= \left(\frac{\partial\Phi}{\partial\phi}\right)^2 R_c^2 \\ G &= c^2 R_c^2 \cos^2\phi \end{aligned}\right\} . \tag{5.1.4}$$

The condition for conformality, as given by (2.8.2), and applied to the two corresponding orthogonal parametric systems is

$$\frac{E}{e} = \frac{G}{g} = m^2 \tag{5.1.5}$$

where m^2 is a constant.

The fundamental transformation matrix (2.7.11) gives, with the aid of (5.1.5)

$$\left.\begin{aligned} E &= \left(\frac{\partial\Phi}{\partial\phi}\right)^2 E' + \left(\frac{\partial\Lambda}{\partial\phi}\right)^2 G' = e m^2 \\ O &= \left(\frac{\partial\Phi}{\partial\phi}\right)\left(\frac{\partial\Phi}{\partial\lambda}\right) E' + \left(\frac{\partial\Lambda}{\partial\phi}\right)\left(\frac{\partial\Lambda}{\partial\lambda}\right) G' \\ G &= \left(\frac{\partial\Phi}{\partial\lambda}\right)^2 E' + \left(\frac{\partial\Lambda}{\partial\lambda}\right)^2 G' = g m^2 \end{aligned}\right\} . \tag{5.1.6}$$

From (5.1.3)

$$\frac{\partial\Phi}{\partial\lambda} = \frac{\partial\Lambda}{\partial\phi} = 0 . \tag{5.1.7}$$

Substitute (5.1.7) into (5.1.6).

$$\left.\begin{aligned} E &= \left(\frac{\partial\Phi}{\partial\phi}\right)^2 E' = e m^2 \\ G &= \left(\frac{\partial\Lambda}{\partial\lambda}\right)^2 G' = g m^2 \end{aligned}\right\} . \tag{5.1.8}$$

Write (5.1.8) as a proportion to eliminate m^2. This is a form of the condition of conformality.

$$\frac{\left(\dfrac{\partial \Phi}{\partial \phi}\right)}{e} \; E' = \frac{\left(\dfrac{\partial \Lambda}{\partial \lambda}\right)^2}{g} \; G' . \tag{5.1.9}$$

Substitute (5.1.1), (5.1.2), and the partial derivative of (5.1.3) into (5.1.9).

$$\frac{1}{R_m^2} \left(\frac{\partial \Phi}{\partial \phi}\right)^2 R_c^2 = \frac{c^2 R_c^2 \cos^2 \Phi}{R_p^2 \cos^2 \phi} = m^2 . \tag{5.1.10}$$

Convert (5.1.10) into an ordinary differential equation, and take the square root. Then, by separating the variables,

$$\frac{d\Phi}{\cos \Phi} = \frac{c R_m}{R_p \cos \phi} \; d\phi . \tag{5.1.11}$$

Substitute (3.2.9) and (3.2.16) into (5.1.11) to obtain

$$\frac{d\Phi}{\cos \Phi} = \frac{\dfrac{ca(1 - e^2)}{(1 - e^2 \sin^2 \phi)^{3/2}}}{\dfrac{a \cos \phi}{(1 - e^2 \sin^2 \phi)^{1/2}}} \; d\phi$$

$$= \frac{c(1 - e^2)}{(1 - e^2 \sin^2 \phi) \cos \phi} .$$

The solution of this differential equation is

$$\ln \tan \left(\frac{\pi}{4} + \frac{\Phi}{2}\right) = c \ln \tan \left(\frac{\pi}{4} + \frac{\phi}{2}\right)\left(\frac{1 - e \sin \phi}{1 + e \sin \phi}\right)^{e/2} + K . \tag{5.1.12}$$

The constant K is removed by requiring that Φ and ϕ are coincidently equal to zero. Thus, from (5.1.12)

$$\tan \left(\frac{\pi}{4} + \frac{\Phi}{2}\right) = \left\{ \ln \left(\frac{\pi}{4} + \frac{\phi}{2}\right)\left(\frac{1 - e \sin \phi}{1 + e \sin \phi}\right)^{e/2} \right\}^c . \tag{5.1.13}$$

Note that this integral was encountered before, in Section 3.3, for the loxodromic curve on the spheroid.

In order that Λ and λ coincide at zero, from the second of (5.1.3), $c_1 = 0$, and

$$\Lambda = c\lambda .$$

It remains to find the value of the constant c for the particular transformation from the spheroid to the conformal sphere.

Consider a Taylor's series expansion of the constant m^2 about the origin. By origin we mean in this development, the latitude selected as the origin of the map. Recall that the partial derivatives of m^2 with respect to λ are zero. Then,

$$m^2 = m_0^2 + \left(\frac{\partial m^2}{\partial \phi}\right)_0 \Delta\phi + \frac{1}{2}\left(\frac{\partial^2 m^2}{\partial \phi^2}\right)_0 (\Delta\phi)^2 + \ldots . \tag{5.1.15}$$

Also from (5.1.10)

$$m = \frac{c\, R_c \cos \Phi_c}{R_p \cos \phi} . \tag{5.1.16}$$

At the origin of the map, $m_0 = 1$, by definition of the conformal projection. This aspect of map projections will be explored in Chapter 7 on the theory of distortion. Considering (5.1.6) at the origin,

$$m_0 = 1 = \frac{c\, R_c \cos \Phi_0}{R_p \cos \phi_0} . \tag{5.1.17}$$

Let

$$\frac{\partial m}{\partial \phi} = c\, R_c \left\{\frac{\partial}{\partial \phi}\left[\frac{\cos \Phi}{R_p \cos \phi}\right]\right\} = 0 . \tag{5.1.18}$$

Taking the derivative of the portion of (5.1.6) in brackets, and substituting (3.2.9) in this

$$-\frac{\sin \Phi}{R_p \cos \phi}\frac{\partial \Phi}{\partial \phi} - \frac{\cos \Phi}{(R_p \cos \phi)^2}\left(-R_p \sin \phi + \cos \phi \frac{\partial R_p}{\partial \phi}\right) = 0$$

$$\sin \Phi \frac{\partial \Phi}{\partial \phi} = -\frac{\cos \Phi}{R_p \cos \phi}\left|-R_p \sin \phi + \frac{\cos \phi\, a e^2 \sin \phi \cos \phi}{(1 - e^2 \sin^2 \phi)^{3/2}}\right|$$

$$= -\frac{\cos \Phi \sin \phi}{R_p \cos \phi}\left|-R_p - \frac{a e^2 \cos^2 \phi}{(1 - e^2 \sin^2 \phi)^{3/2}}\right|$$

$$= \frac{\cos\Phi \sin\phi}{R_p \cos\phi}\left[\frac{a(1-e^2\sin^2\phi) - ae^2\cos^2\phi}{(1-e^2\sin^2\phi)^{3/2}}\right]$$

$$= \frac{\cos\Phi \sin\phi}{R_p \cos\phi}\left[\frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{3/2}}\right]. \qquad (5.1.19)$$

Substitute (3.2.16) into (5.1.19).

$$\sin\Phi \ \frac{\partial\Phi}{\partial\phi} = \cos\Phi \ \frac{R_m \sin\phi}{R_p \cos\phi}. \qquad (5.1.20)$$

Evaluate (5.1.20) at the origin of the map.

$$\sin\Phi_0 \left(\frac{\partial\Phi}{\partial\phi}\right)_0 = \cos\Phi_0 \ \frac{R_{m0}\sin\phi_0}{R_{p0}\cos\phi_0}. \qquad (5.1.21)$$

Substitute (5.1.10) into (5.1.21).

$$\frac{\sin\Phi_0 \cos\Phi_0 \, c \, R_{m0}}{R_{p0}\cos\phi_0} = \frac{\cos\Phi_0 \, R_{m0}\sin\phi_0}{R_{p0}\cos\phi_0}$$

$$\sin\phi_0 = c\sin\Phi_0 . \qquad (5.1.22)$$

The next step is to obtain the second partial derivative of m, and equate this to zero. This is accomplished by using (5.1.23) to obtain

$$\tan\phi_0 = \sqrt{\frac{R_{p0}}{R_{m0}}} \ \tan\Phi_0 . \qquad (5.1.23)$$

From (5.1.22)

$$\sin\Phi_0 = \frac{\sin\phi_0}{c} \qquad (5.1.24)$$

$$\cos\Phi_0 = \sqrt{1 - \frac{\sin^2\phi_0}{c^2}} . \qquad (5.1.25)$$

Substitute (5.1.24) and (5.1.25) into (5.1.23). Also, substitute (3.2.9) and (3.2.16).

$$\frac{\sin\phi_0}{\cos\phi_0} = \sqrt{\frac{R_{p0}}{R_{m0}}}\,\frac{\sin\Phi_0}{\cos\Phi_0} = \sqrt{\frac{\dfrac{a}{(1-e^2\sin^2\phi_0)^{1/2}}}{\dfrac{a(1-e^2)}{(1-e^2\sin^2\phi_0)^{3/2}}}}\cdot\frac{\dfrac{\sin\phi_0}{c}}{\sqrt{1-\dfrac{\sin^2\phi_0}{c}}}$$

$$\frac{1}{\cos\phi_0} = \sqrt{\frac{1-e^2\sin^2\phi_0}{1-e^2}}\;\frac{1}{\sqrt{c^2-\sin^2\phi_0}}$$

$$\frac{1}{\cos^2\phi_0} = \frac{1-e^2\sin^2\phi_0}{1-e^2}\cdot\frac{1}{c^2-\sin^2\phi_0}$$

$$\cos^2\phi_0 = \frac{(1-e^2)(c^2-\sin^2\phi_0)}{1-e^2\sin^2\phi_0}$$

$$\cos^2\phi_0(1-e^2\sin^2\phi_0) = (1-e^2)(c^2-\sin^2\phi_0)$$

$$\cos^2\phi_0 - e^2\cos^2\phi_0\sin^2\phi_0 = (1-e^2)c^2 - \sin^2\phi_0 + e^2\sin^2\phi_0$$

$$\cos^2\phi_0 + \sin^2\phi_0 - e^2\sin^2\phi_0(1+\cos^2\phi_0) = c^2(1-e^2)$$

$$1 - e^2(1-\cos^2\phi_0)(1+\cos^2\phi_0) = c^2(1-e^2)$$

$$1 + e^2(\cos^4\phi_0 - 1) = c^2(1-e^2)$$

$$c^2 = \frac{1-e^2+e^2\cos^4\phi_0}{1-e^2}$$

$$c = \left(1 + \frac{e^2\cos^4\phi_0}{1-e^2}\right)^{1/2}. \qquad (5.1.26)$$

The radius of the conformal sphere can be found from (5.1.23)

$$\frac{\sin\phi_0}{\cos\phi_0} = \sqrt{\frac{R_{p0}}{R_{m0}}}\,\frac{\sin\Phi_0}{\cos\Phi_0}. \qquad (5.1.27)$$

From (5.1.24)

$$\frac{c \sin \Phi_0}{\cos \phi_0} = \sqrt{\frac{R_{p0}}{R_{m0}}}\ \frac{\sin \Phi_0}{\cos \Phi_0}, \quad c = \frac{\cos \phi_0}{\cos \Phi_0}\ \frac{R_{p0}}{R_{m0}}. \tag{5.1.28}$$

Eliminate c between (5.1.17) and (5.1.28).

$$1 = \frac{R_c}{\sqrt{R_{p0}\,R_{m0}}}$$

$$R_c = \sqrt{R_{p0}\,R_{m0}}\,. \tag{5.2.29}$$

Equations (5.1.13), (5.1.14), and (5.1.26) can be used to convert from the spheroidal earth to a conformally equivalent sphere, with a radius given by equation (5.1.29). Once this is done, the conformal projection from the conformal sphere is relatively easy. Table 5.1.1 gives the conformal latitude in terms of geodetic latitude for the WGS-72 ellipsoid when ϕ_0 is arbitrarily chosen as 0°. Note that, unlike the development for authalic latitude, the conformal sphere depends on a particular choice of origin, ϕ_0. Note also that (5.1.29), or the radius of the conformal sphere is also dependent on this origin. Thus, the radius of the conformal sphere contracts or expands as the choice of the origin dictates.

Table 5.1.1. Conformal Latitude as a Function of Geodetic Latitude for the WGS-72 Spheroid.

Geodetic Latitude	Conformal Latitude
0.0000	0.0000
5.0000	6.5050
10.0000	13.0117
15.0000	19.5215
20.0000	26.0362
25.0000	32.5571
30.0000	39.0855
35.0000	45.6224
40.0000	52.1380
45.0000	58.7257
50.0000	65.2933
55.0000	71.8719
60.0000	78.4619
65.0000	85.0632

$\phi_0 = 0°$

5.2 Mercator Projection [8], [20], [22], [23], [24]

The Mercator projection, devised in 1569 by Gerhard Kramer, whose Latin name was Mercator, is the classic of modern map projections. It was derived as an aid to navigation in the initial days of the age of ocean exploration, and has continued its utility through the age of space exploration. The regular, or equatorial, Mercator projection, with its areas of lesser distortion north and south of the equator, and including the major maritime trade routes was and is a natural vehicle for ocean navigation. Transverse Mercator projections, with the Lambert conformal, are the backbone of the quadrangle system for topographic surveying. Oblique Mercator projections, with the line of zero distortion along the nominal satellite re-entry footprint have been used in the recovery charts for the Mercury, Gemini, and Apollo missions.

All three of the Mercator variations, the regular, the oblique, and the transverse, will be considered in terms of a double transformation, that is from the spheroid to the conformal sphere, and then to the mapping surface, and the equatorial in terms of a direct transformation of the spheroid to the map.

The Mercator projection entails, in both approaches, a transformation from the spheroid to a cylinder. The Mercator can be considered in terms of a semi-graphical technique. One can, with extreme patience, construct a Mercator projection by a graphical means. In fact, Mercator, before the development of calculus, did just that. The objection to this is that there is a varying projection point. Calculations are needed to locate the point of emanation of the projection ray. This is shown in Figure 5.2.1. For each and every latitude, a different point is needed as the origin of an interior ray which intersect both the surface of the spheroid (or conformal sphere) and the developable surface, the cylinder. Thus, the reasonable approach is to use a mathematical method.

The regular Mercator projection will be developed first for the conformal sphere. Then, the rotation formulas of Section 2.10 will be applied to produce the oblique and transverse cases.

For the regular Mercator projection, let the Cartesian mapping coordinates be given by the functional relations.

$$\left.\begin{aligned} x &= x(\lambda) \\ y &= y(\phi) \end{aligned}\right\} . \tag{5.2.1}$$

In particular, the first function is taken as a linear combination

$$x = aS(\lambda - \lambda_0) \tag{5.2.2}$$

where a is the radius of the conformal sphere, and S is the scale factor.

The elemental forms of the second of (5.2.1), and (5.2.2) are

$$\begin{aligned} dx &= aS\, d\lambda \\ dy &= \frac{dy}{d\phi}\, d\phi . \end{aligned} \tag{5.2.3}$$

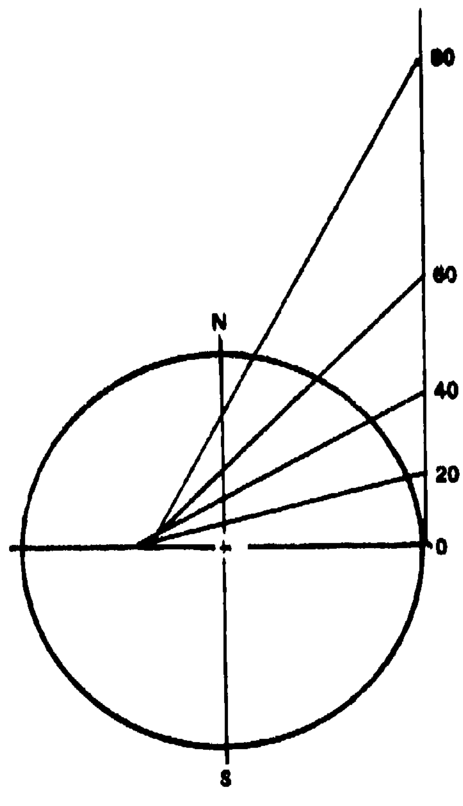


Figure 5.2.1. The varying projection point of the equatorial Mercator projection

The first fundamental form of the plane is

$$(ds)^2 = (dy^2) + (dx)^2 . \tag{5.2.4}$$

Substitute (5.2.3) into (5.2.4).

$$(ds)^2 = \left(\frac{dy}{d\phi}\right)^2 d\phi^2 + a^2 S^2 (d\lambda)^2 . \tag{5.2.5}$$

The first fundamental quantities of (5.2.5) are

$$\left.\begin{aligned} E &= \left(\frac{dy}{d\phi}\right)^2 \\ G &= a^2 S^2 \end{aligned}\right\} . \tag{5.2.6}$$

The first fundamental form of the conformal sphere is

$$(ds)^2 = a^2 (d\phi)^2 + a^2 \cos^2 \phi \, (d\lambda)^2 . \tag{5.2.7}$$

The first fundamental quantities, from (5.2.7), are

$$\left.\begin{aligned} e &= a^2 \\ g &= a^2 \cos^2 \phi \end{aligned}\right\} . \tag{5.2.8}$$

For the orthogonal systems of the plane and the conformal sphere, the relation of conformality, from (2.8.2) is

$$\frac{E}{e} = \frac{G}{g} . \tag{5.2.9}$$

Substituting (5.2.6) and (5.2.8) into (5.2.9).

$$\frac{a^2 S^2}{a^2 \cos^2 \phi} = \frac{\left(\frac{dy}{d\phi}\right)^2}{a^2}$$

$$\frac{dy}{d\phi} = \frac{aS}{\cos \phi}$$

$$y = a \int \frac{S \, d\phi}{\cos \phi} = aS \ln \tan \left(\frac{\pi}{4} + \frac{\phi}{2}\right) + c . \tag{5.2.11}$$

In (5.2.10), choose c such that y = 0 when $\phi = 0$. Then, c = 0.

$$y = aS \ln \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right). \tag{5.2.11}$$

Equations (5.2.2) and (5.2.11) provide the transformation from the conformal sphere to the cylinder, or the map. The oblique and transverse Mercator projections will now be gained from the equatorial Mercator by utilization of the rotation formulas of Section 2.10. Substitute (2.10.4) and (2.10.5) into (5.2.2) and (5.2.11) to obtain the oblique Mercator projection.

$$\begin{aligned} x &= aS\alpha \\ &= aS \tan^{-1}\left\{\frac{\sin(\lambda - \lambda_p)}{\cos\phi_p \tan\phi - \sin\phi_p \cos(\lambda - \lambda_p)}\right\} \end{aligned} \tag{5.2.12}$$

where ϕ_p is the latitude, and λ_p is the longitude of the pole of the reference plane.

$$\begin{aligned} y &= aS \ln \tan\left(\frac{\pi}{4} + \frac{h}{2}\right) \\ &= \frac{aS}{2} \ln\left(\frac{1 + \sin h}{1 - \sin h}\right) \end{aligned} \tag{5.2.13}$$

$$y = \frac{aS}{2} \ln\left[\frac{1 + \sin\phi \sin\phi_p + \cos\phi \cos\phi_p \cos(\lambda - \lambda_p)}{1 - \sin\phi \sin\phi_p - \cos\phi \cos\phi_p \cos(\lambda - \lambda_p)}\right]. \tag{5.2.14}$$

The transverse Mercator projection is obtained as a special case of the oblique Mercator projection by letting $\phi_p = 0$ in (5.2.12) and (5.2.14).

$$x = aS \tan^{-1}\left[\frac{\sin(\lambda - \lambda_p)}{\tan\phi}\right] \tag{5.2.15}$$

$$y = \frac{aS}{2} \ln\left[\frac{1 + \cos\phi \cos(\lambda - \lambda_p)}{1 - \cos\phi \cos(\lambda - \lambda_p)}\right]. \tag{5.2.16}$$

Figures 5.2.2, 5.2.3, and 5.2.4 are specimens of the equatorial, oblique, and transverse Mercator projections, respectively. Plotting tables for these projections are given in Tables 5.2.1, 5.2.2, and 5.2.3, respectively. Note that, in the equatorial Mercator, the parallels and meridians are straight lines, intersecting at right angles. This means that the convergency of the meridians does not occur, and distortion becomes excessive in a poleward direction. In fact, the point of the pole is approaching infinity. Thus, the equatorial projection is useless at extremely high and low latitudes. In the transverse Mercator, the central meridian and the equator are the only straight lines. However, curved meridians and parallels intersect orthogonally. For the oblique Mercator, there are no straight meridians or parallels. However, orthogonality is present.

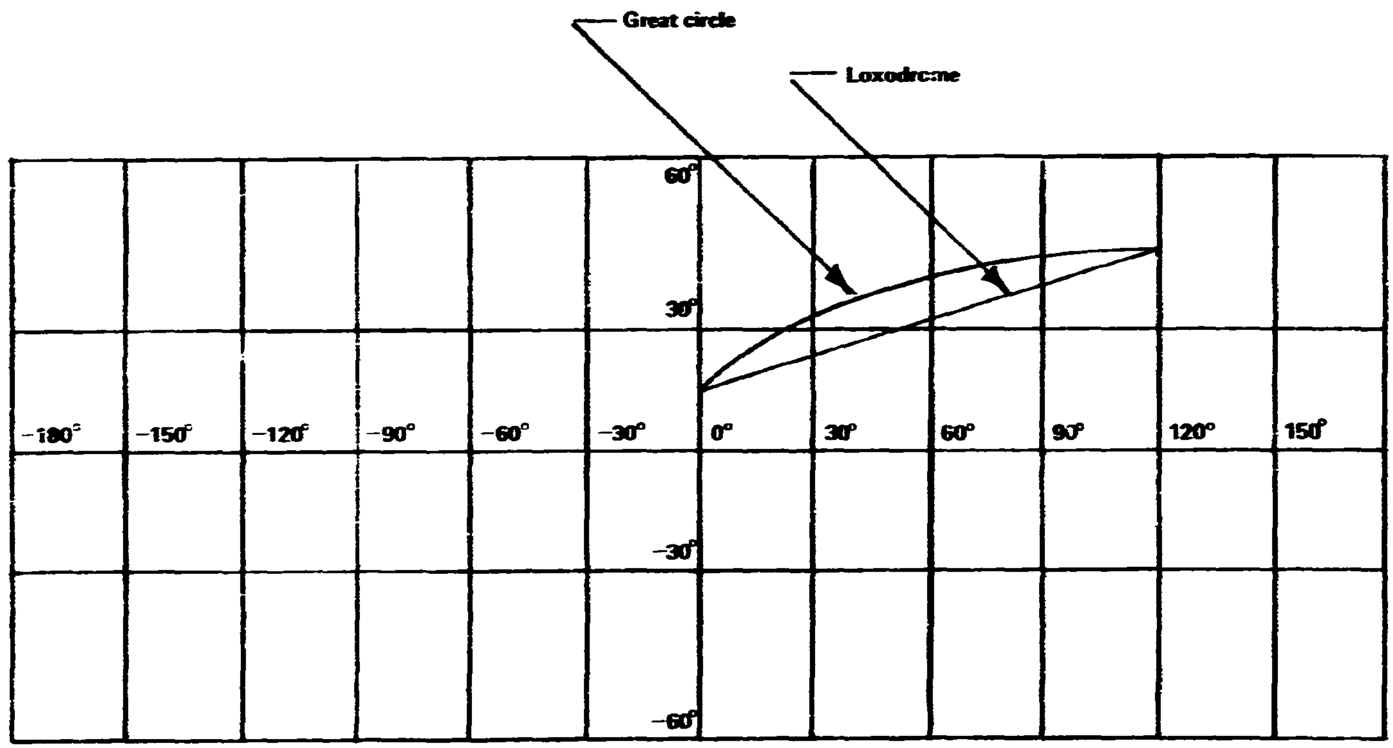


Figure 5.2.2. Regular or equatorial Mercator projection

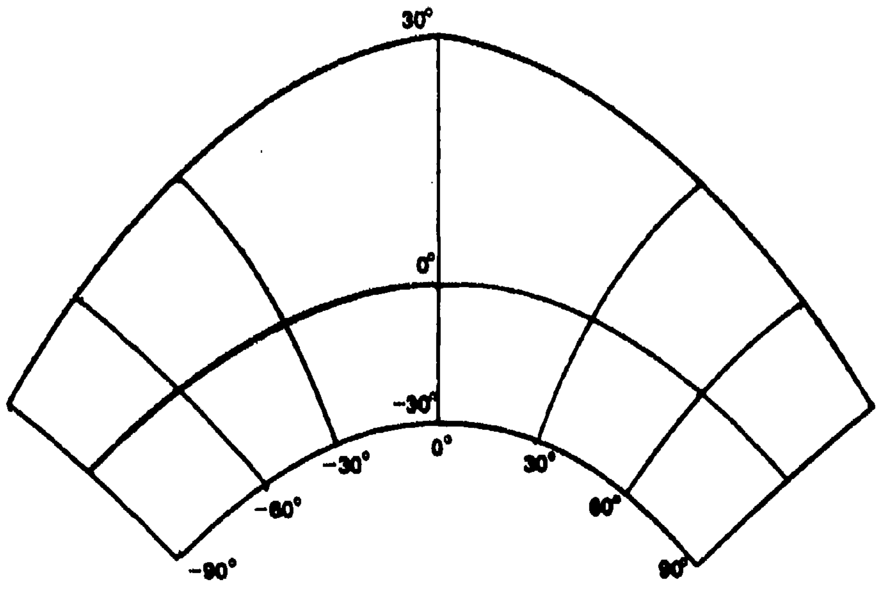


Figure 5.2.3. Oblique Mercator projection

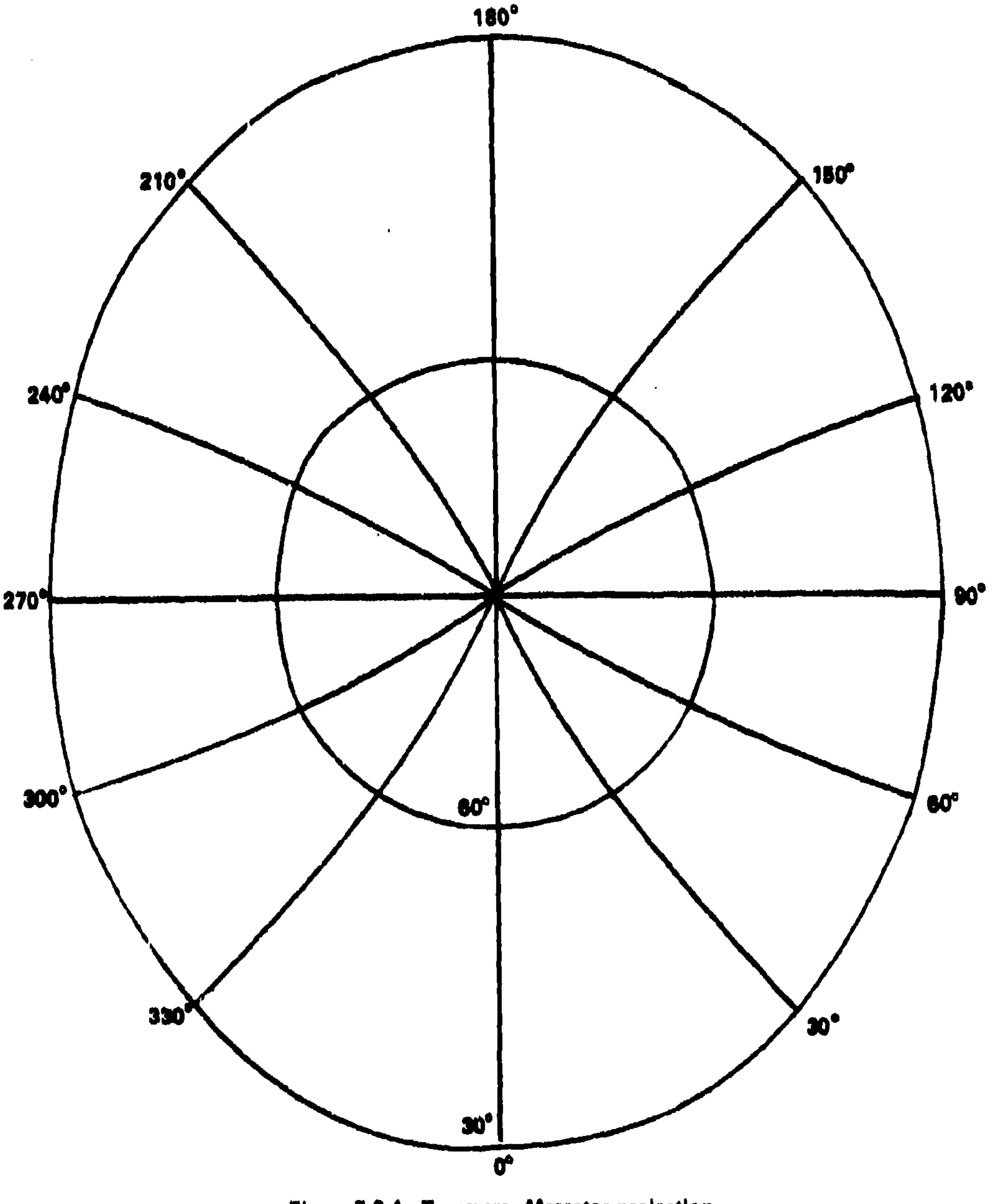


Figure 5.2.4. Transverse Mercator projection

Table 5.2.1. Equatorial Mercator Projection.

Regular Mercator

Latitude*	Longitude*	X**	Y**
0.0000	0.0000	0.000	-.000
0.0000	30.0000	3.340	-.000
0.0000	60.0000	6.679	-.000
0.0000	90.0000	10.019	-.000
0.0000	120.0000	13.359	-.000
0.0000	150.0000	16.698	-.000
0.0000	180.0000	20.038	-.000
30.0000	0.0000	0.000	3.482
30.0000	30.0000	3.340	3.482
30.0000	60.0000	6.679	3.482
30.0000	90.0000	10.019	3.482
30.0000	120.0000	13.359	3.482
30.0000	150.0000	16.698	3.482
30.0000	180.0000	20.038	3.482
60.0000	0.0000	0.000	8.363
60.0000	30.0000	3.340	8.363
60.0000	60.0000	6.679	8.363
60.0000	90.0000	10.019	8.363
60.0000	120.0000	13.359	8.363
60.0000	150.0000	16.698	8.363
60.0000	180.0000	20.038	8.363

$\phi_0 = 90^\circ$ *Degrees
$\lambda_0 = 0^\circ$ **Meters

Table 5.2.2. Oblique Mercator Projection.
Mercator Oblique Case

Latitude*	Longitude*	X**	Y**
-30.0000	0.0000	0.000	1.689
-30.0000	15.0000	1.485	1.547
-30.0000	30.0000	2.905	1.139
-30.0000	45.0000	4.219	.508
-30.0000	60.0000	5.417	-.302
-30.0000	75.0000	6.515	-1.260
-30.0000	90.0000	7.547	-2.357
-15.0000	0.0000	0.000	3.503
-15.0000	15.0000	1.839	3.309
-15.0000	30.0000	3.556	2.767
-15.0000	45.0000	5.090	1.974
-15.0000	60.0000	6.447	1.019
-15.0000	75.0000	7.671	-.040
-15.0000	90.0000	8.825	-1.181
0.0000	0.0000	0.000	5.621
0.0000	15.0000	2.310	5.324
0.0000	30.0000	4.367	4.546
0.0000	45.0000	6.093	3.503
0.0000	60.0000	7.547	2.357
0.0000	75.0000	8.825	1.180
0.0000	90.0000	10.019	-.000
15.0000	0.0000	0.000	8.400
15.0000	15.0000	3.081	7.849
15.0000	30.0000	5.541	6.579
15.0000	45.0000	7.380	5.125
15.0000	60.0000	8.825	3.716
15.0000	75.0000	10.061	2.402
15.0000	90.0000	11.213	1.181
30.0000	0.0000	0.000	12.932
30.0000	15.0000	4.819	11.375
30.0000	30.0000	7.547	8.886
30.0000	45.0000	9.196	6.776
30.0000	60.0000	10.421	5.054
30.0000	75.0000	11.480	3.607
30.0000	90.0000	12.491	2.357

$\phi_0 = 45^\circ$ *Degrees
$\lambda_0 = 0^\circ$ **Meters

Table 5.2.3. Transverse Mercator Projection.
Mercator Transverse Case

Latitude*	Longitude*	X**	Y**
30.0000	0.0000	0.000	-8.400
30.0000	15.0000	2.688	-7.714
30.0000	30.0000	4.552	-6.206
30.0000	45.0000	5.652	-4.546
30.0000	60.0000	6.268	-2.957
30.0000	75.0000	6.583	-1.454
30.0000	90.0000	6.679	.000
45.0000	0.0000	0.000	-5.621
45.0000	15.0000	1.615	-5.324
45.0000	30.0000	2.957	-4.546
45.0000	45.0000	3.926	-3.503
45.0000	60.0000	4.552	-2.357
45.0000	75.0000	4.899	-1.180
45.0000	90.0000	5.009	.000
60.0000	0.0000	0.000	-3.503
60.0000	15.0000	.946	-3.360
60.0000	30.0000	1.792	-2.957
60.0000	45.0000	2.472	-2.357
60.0000	60.0000	2.957	-1.629
60.0000	75.0000	3.245	-.830
60.0000	90.0000	3.339	.000
75.0000	0.0000	0.000	-1.689
75.0000	15.0000	.442	-1.629
75.0000	30.0000	.849	-1.454
75.0000	45.0000	1.194	-1.180
75.0000	60.0000	1.454	-.830
75.0000	75.0000	1.615	-.428
75.0000	90.0000	1.670	.000
90.0000	0.0000	0.037	.000
90.0000	15.0000	0.037	.000
90.0000	30.0000	0.037	.000
90.0000	45.0000	0.037	.000
90.0000	60.0000	0.037	.000
90.0000	75.0000	0.037	.000
90.0000	90.0000	0.037	-.000

$\phi_0 = 0°$ *Degrees
$\lambda_0 = 0°$ **Meters

In Figure 5.2.2, the loxodrome (or rhumbline) and the great circle are portrayed on a equatorial Mercator projection. The loxodrome is a line which intersects successive meridians at the same azimuth, or bearing angle. On the Mercator projection the loxodrome is a straight line, and the great circle (or geodesic) is a curved line. The gnomonic projection of Section 6.1 has the reverse of this situation. As will be seen, the gnomonic projection has great circles as straight lines, and the loxodromes are curved lines. Thus, by using the Mercator and the gnomonic projections together, one can build a series of bearings which will approximate, piecewise, a great circle route. This combines ease of navigation with an approximation to the shortest distance between two arbitrary points. This method is used for both maritime and aerial navigation.

The Mercator projection, as mentioned above, can also be derived by a direct transformation from the spheroid to the plotting surface. This will now be done to compare the labor in these two approaches.

From (3.3.1), an element of distance along a parallel of the spheroid is

$$dp = \frac{a \cos\phi \, d\lambda}{(1 - e^2 \sin^2\phi)^{1/2}} .$$

The infinitesimal distance along the parallel of the map is $a d\lambda$. Thus, the scale along the parallel is

$$\frac{dp}{a d\lambda} = \frac{\cos\phi}{(1 - e^2 \sin^2\phi)^{1/2}} . \tag{5.2.17}$$

From (3.2.16), an element of distance along the meridian is given by

$$dm = \frac{a(1 - e^2)\, d\phi}{(1 - e^2 \sin^2\phi)^{3/2}} . \tag{5.2.18}$$

Let dy be the element of distance on the meridian of the map which represents the elemental distance dm along the meridianal ellipse. The ratio of dm to dy must equal the scale along the parallel, if conformality is to be maintained. Thus, from (5.2.17) and (5.2.18).

$$\frac{dm}{dy} = \frac{a(1 - e^2)\, d\phi}{dy\,(1 - e^2 \sin^2\phi)^{3/2}} = \frac{\cos\phi}{(1 - e^2 \sin^2\phi)}$$

$$dy = \frac{a(1 - e^2)\, d\phi}{(1 - e^2 \sin^2\phi)\cos\phi} . \tag{5.2.19}$$

The distance of the parallel of latitude, as measured along a meridian, from the equator, is found by integrating (5.2.19). This is done by expanding the integral in partial fractions.

$$y = \int_0^{\phi} \frac{a(1-e^2)\,d\phi}{(1-e^2\sin^2\phi)\cos\phi}$$

$$= a\left\{\int_0^{\phi}\frac{d\phi}{\cos\phi} + \frac{e}{2}\int_0^{\phi}\frac{-e\cos\phi\,d\phi}{1-e\sin\phi} - \frac{e}{2}\int_0^{\phi}\frac{e\cos\phi\,d\phi}{1+e\sin\phi}\right\}$$

$$= a\left\{\int_0^{\phi}\frac{d\phi}{\sin\left(\frac{\pi}{2}+\phi\right)} + \frac{e}{2}\int_0^{\phi}\frac{-e\cos\phi\,d\phi}{1-e\sin\phi} - \frac{e}{2}\int_0^{\phi}\frac{e\cos\phi\,d\phi}{1-e\sin\phi}\right\}$$

$$= a\left\{\int_0^{\phi}\frac{\cos\left(\frac{\pi}{4}+\frac{\phi}{2}\right)}{\sin\left(\frac{\pi}{4}+\frac{\phi}{2}\right)}\frac{d\phi}{2} - \int_0^{\phi}\frac{\sin\left(\frac{\pi}{4}+\frac{\phi}{2}\right)}{\cos\left(\frac{\pi}{4}+\frac{\phi}{2}\right)}\frac{d\phi}{2}\right.$$

$$\left. + \frac{e}{2}\int_0^{\phi}\frac{-e\cos\phi\,d\phi}{1-e\sin\phi} - \frac{e}{2}\int_0^{\phi}\frac{e\cos\phi\,d\phi}{1+e\sin\phi}\right\}$$

$$y = a\left\{\ln\left[\sin\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right] - \ln\left[\cos\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right]\right.$$

$$\left. + \frac{e}{2}\ln(1-e\sin\phi) - \frac{e}{2}\ln(1+e\sin\phi)\Bigg]\right\}$$

$$= a\left\{\ln\left[\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right] + \frac{e}{2}\ln\left[\left(\frac{1-e\sin\phi}{1+e\sin\phi}\right)\right]\right.$$

$$= a\ln\left[\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1-e\sin\phi}{1+e\sin\phi}\right)^{e/2}\right\}. \qquad (5.2.20)$$

Thus, we have in (5.2.20) the same results as (5.2.14) if e is set equal to zero.

The distance along the equator can be found from the integral

$$x = a\int_0^{\lambda} d\lambda$$

$$= a\Delta\lambda. \qquad (5.2.21)$$

The amount of labor in either the direct or the indirect method of transformation is significant. The amount of computer time consumed in evaluating either set of equations is similar. The rotations of Section 2.10 can be applied to (5.2.20) and (5.2.21) to obtain approximations to the oblique and transverse Mercator projections.

5.3 Lambert Conformal [20], [23]

The Lambert conformal projection is a projection from the spheroidal earth onto a cone, which serves as the developable surface. This can be done rather simply by transforming from the conformal sphere to the cone. The transformation can also be accomplished directly from the spheroid to the cone. This second approach will be followed in this section for Lambert conformal projections with one and two standard parallels. Then, the eccentricity is set equal to zero to accommodate transformations from the conformal sphere.

For the Lambert conformal projection with one standard parallel, the conical mapping surface is tangent to the spheroid at this standard parallel. The axis of the cone coincides with the rotation, or polar axis of the earth. The meridians are straight lines converging at the apex of the cone. One of these meridians is arbitrarily chosen as the central meridian, λ_0. The parallels are a set of concentric circles.

The polar coordinates of a point P are ρ and θ. The Cartesian coordinates of this same point are

$$\left.\begin{aligned} x &= \rho \sin\theta \cdot S \\ y &= (\rho_0 - \rho\cos\theta)S \end{aligned}\right\} \qquad (5.3.1)$$

where ρ_0 is the radius vector from the apex of the cone to the circle of tangency.

Again, the elemental distance on the spheroid is found from the first fundamental form

$$(ds)^2 = R_m^2\,(d\phi)^2 + R_p^2\cos^2\phi\,(d\lambda)^2$$

with fundamental quantities

$$\left.\begin{aligned} e &= R_m^2 \\ g &= R_p^2\cos^2\phi \end{aligned}\right\}. \qquad (5.3.2)$$

On the conical surface, the first fundamental form is

$$(ds)^2 = (d\rho)^2 + \rho^2\,(d\theta)^2$$

with fundamental quantities

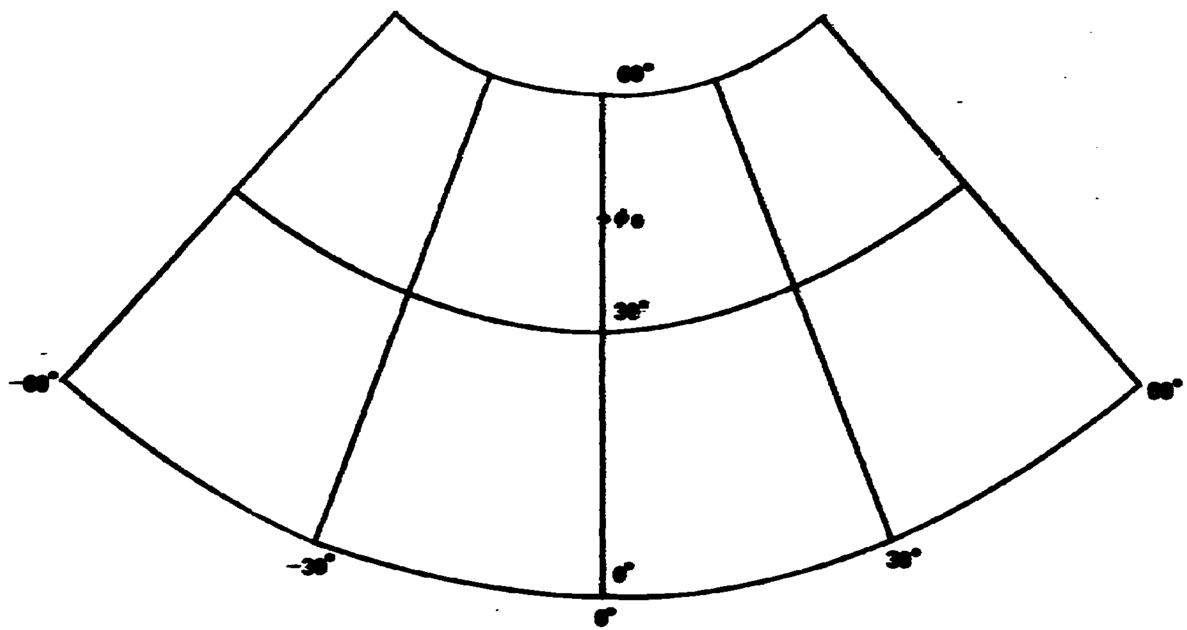


Figure 5.3.1. Lambert conformal projection, one standard parallel

$$\left.\begin{aligned} E' &= 1 \\ G' &= \rho^2 \end{aligned}\right\} . \tag{5.3.3}$$

Two conditions will be imposed. One is that

$$\rho = \rho(\phi) \tag{5.3.4}$$

and the second is that

$$\theta = c_1 \lambda + c_2 . \tag{5.3.5}$$

From (5.3.5)

$$\frac{\partial \theta}{\partial \lambda} = c_1 . \tag{5.3.6}$$

From the fundamental transformation matrix, (2.7.11)

$$\left.\begin{aligned} E &= \left(\frac{\partial \rho}{\partial \phi}\right)^2 E' \\ G &= \left(\frac{\partial \theta}{\partial \lambda}\right)^2 E' \end{aligned}\right\} . \tag{5.3.7}$$

Substitute (5.3.3) into (5.3.7)

$$\left.\begin{aligned} E &= \left(\frac{\partial \rho}{\partial \phi}\right)^2 \\ G &= \left(\frac{\partial \theta}{\partial \lambda}\right)^2 \rho^2 \end{aligned}\right\} . \tag{5.3.8}$$

Substitute (5.3.6) into the second of (5.3.8).

$$G = c_1^2 \rho^2 . \tag{5.3.9}$$

From the condition of conformality (5.1.9) for two orthogonal systems.

$$\frac{\left(\frac{\partial \rho}{\partial \phi}\right)^2 E'}{e} = \frac{\left(\frac{\partial \theta}{\partial \lambda}\right)^2 G'}{g} = m^2 . \tag{5.3.10}$$

Substitute (5.3.2), (5.3.3), and (5.3.6) into (5.3.10).

$$\frac{\left(\frac{\partial\rho}{\partial\phi}\right)^2}{R_p^2} = \frac{c_1^2\rho^2}{R_p^2\cos^2\phi} = m^2 . \tag{5.3.11}$$

Take the square root of (5.3.11), and convert the result to an ordinary differential equation

$$\frac{d\rho}{\rho} = -\frac{R_m c_1}{R_p \cos\phi}\, d\phi . \tag{5.3.12}$$

The minus sign is chosen since ρ decreases as ϕ increases.

Equation (5.3.12) can be integrated by the method of Section 5.2 for the Mercator projection to obtain

$$\ln\rho = -c_1 \ln\left\{\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1-e\sin\phi}{1+e\sin\phi}\right)^{e/2}\right\} + \ln c_3$$

$$\rho = c_3\left\{\tan\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\left(\frac{1+e\sin\phi}{1-e\sin\phi}\right)^{e/2}\right\}^{c_1} . \tag{5.3.13}$$

The constants c_1, c_2, and c_3 must be evaluated now. First, from (5.3.5), it is required that $\theta = 0$, when $\lambda = 0$. Thus, $c_2 = 0$.

Next, consider c_3. At the origin of the Cartesian coordinate system of the map (ϕ_0, λ_0), the cone is tangent to the spheroid. Thus, similar to the development of Section 1.6

$$\rho_0 = R_{p0}\cot\phi_0 . \tag{5.3.14}$$

Evaluate (5.3.13) at ϕ_0, and equate to (5.3.14).

$$R_{p0}\cot\phi_0 = c_3\left\{\tan\left(\frac{\pi}{4}-\frac{\phi_0}{2}\right)\left(\frac{1+e\sin\phi_0}{1-e\sin\phi_0}\right)^{e/2}\right\}^{c_1}$$

$$c_3 = \frac{R_{p0}\cot\phi_0}{\left\{\tan\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\left(\frac{1+e\sin\phi_0}{1-e\sin\phi_0}\right)^{e/2}\right\}^{c_1}} . \tag{5.3.15}$$

Finally, from (5.3.11)

$$m = \frac{c_1 \rho}{R_m \cos\phi}. \tag{5.3.16}$$

At the origin, as will be treated in detail in Chapter 7, $m_0 = 1$. This implies that $(\partial m/\partial\phi)_0 = 0$. Differentiate (5.3.4), and evaluate this at the origin.

$$\frac{\partial m}{\partial \phi} = c_1 \left(\frac{\partial \rho}{\partial \phi}\right) \frac{1}{R_p \cos\phi} + c_1 \rho \frac{R_m \sin\theta}{R_p^2 \cos^2\phi} = 0$$

$$c_1 \left(\frac{\partial \rho}{\partial \phi}\right)_0 + \frac{c_1 \rho_0 R_{m0}}{R_{p0} \cos\phi_0} = 0. \tag{5.3.17}$$

From (5.3.12)

$$\left(\frac{\partial \rho}{\partial \phi_0}\right) = -\frac{c_1 \rho_0 R_{m0}}{R_{p0} \cos\phi_0}. \tag{5.3.18}$$

Substitute (5.3.18) into (5.3.17)

$$-\frac{c_1^2 \rho_0 R_{m0}}{R_{p0} \cos\phi_0} + \frac{c_1 \rho_0 R_{m0} \sin\phi_0}{R_{p0} \cos\phi_0} = 0$$

$$c_1 = \sin\phi_0. \tag{5.3.19}$$

Substitute (5.3.19), and (5.3.14), into (5.3.15)

$$c_3 = \frac{\rho_0}{\left\{\tan\left(\frac{\pi}{4} - \frac{\phi_0}{2}\right)\left(\frac{1 + e\sin\phi_0}{1 - e\sin\phi_0}\right)^{e/2}\right\}^{\sin\phi_0}}. \tag{5.3.20}$$

Substitute (5.3.20) into (5.3.13).

$$\rho = \rho_0 \left\{\frac{\tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right)\left(\frac{1 + e\sin\phi}{1 - e\sin\phi}\right)^{e/2}}{\tan\left(\frac{\pi}{4} - \frac{\phi_0}{2}\right)\left(\frac{1 + e\sin\phi_0}{1 - e\sin\phi_0}\right)^{e/2}}\right\}^{\sin\phi_0}. \tag{5.3.21}$$

Substitute (5.3.19) into (5.3.5), and recall that $c_2 = 0$.

$$\theta = \lambda \sin \phi_0 . \tag{5.3.22}$$

Recall from Section 1.6 that $\sin \phi_0$ is the constant of the cone.

Equations (5.3.21) and (5.3.22), in conjunction with (5.3.1) and (5.3.14) give the plotting equations in Cartesian coordinates. Table 5.3.1 is a plotting table for the Lambert conformal projection with one standard parallel based on an origin of $\phi_0 = 45°$, and $\lambda_0 = 0°$.

The Lambert conformal projection with one standard parallel may be converted to a form for the transformation from the conformal sphere by letting $e = 0$ in (5.3.14) and (5.3.21).

$$\rho_0 = a \cot \phi_0 \tag{5.3.23}$$

$$\rho = \rho_0 \left\{ \frac{\tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right)}{\tan\left(\frac{\pi}{4} - \frac{\phi_0}{2}\right)} \right\}^{\sin\phi_0} . \tag{5.3.24}$$

Equations (5.3.23) and (5.3.24) are then used in conjunction with (5.3.1) to produce the plotting equations.

The next step will be to consider the Lambert conformal projection with two standard parallels. This projection has had considerable utility as aircraft navigation charts, and has been used for star charts by the U.S. Air Force. Again, the meridians are straight lines radiating from the apex of the cone, and the parallels of latitude are concentric circles.

Let the two standard parallels be chosen as ϕ_1 and ϕ_2, where $\phi_2 > \phi_1$. Then, from (5.3.16) and (5.3.19)

$$\left.\begin{aligned} m &= \rho_1 \frac{\sin \phi_0}{R_{p1} \cos \phi_1} \\ &= \rho_2 \frac{\sin \phi_2}{R_{p2} \cos \phi_2} \end{aligned}\right\} \tag{5.3.25}$$

$$\frac{\rho_1}{\rho_2} = \frac{R_{p1} \cos \phi_1}{R_{p2} \cos \phi_2} . \tag{5.3.26}$$

Table 5.3.1. Lambert Conformal Projection, One Standard Parallel.
Lambert Conformal, One Standard Parallel

Latitude*	Longitude*	X**	X**
0.0000	0.0000	0.000	-5.486
0.0000	15.0000	2.186	-5.283
0.0000	30.0000	4.297	-4.681
0.0000	45.0000	6.261	-3.701
0.0000	60.0000	8.011	-2.377
0.0000	75.0000	9.488	-.752
0.0000	90.0000	10.640	1.116
15.0000	0.0000	0.000	-3.470
15.0000	15.0000	1.815	-3.302
15.0000	30.0000	3.567	-2.802
15.0000	45.0000	5.198	-1.988
15.0000	60.0000	6.651	-.888
15.0000	75.0000	7.877	.460
15.0000	90.0000	8.834	2.011
30.0000	0.0000	0.000	-1.683
30.0000	15.0000	1.486	-1.545
30.0000	30.0000	2.921	-1.136
30.0000	45.0000	4.256	-.470
30.0000	60.0000	5.446	.431
30.0000	75.0000	6.449	1.535
30.0000	90.0000	7.232	2.805
45.0000	0.0000	0.000	.000
45.0000	15.0000	1.176	.109
45.0000	30.0000	2.312	.473
45.0000	45.0000	3.369	.960
45.0000	60.0000	4.310	1.673
45.0000	75.0000	5.105	2.547
45.0000	90.0000	5.724	3.552
60.0000	0.0000	0.000	1.690
60.0000	15.0000	.865	1.770
60.0000	30.0000	1.700	2.008
60.0000	45.0000	2.477	2.396
60.0000	60.0000	3.170	2.920
60.0000	75.0000	3.754	3.563
60.0000	90.0000	4.210	4.303

$\phi_0 = 45^\circ$ *Degrees
$\lambda_0 = 0^\circ$ **Meters

From (5.3.21)

$$\frac{\rho_1}{\rho_2} = \left\{ \frac{\tan\left(\frac{\pi}{4} - \frac{\phi_1}{2}\right)\left(\frac{1 + e\sin\phi_1}{1 - e\sin\phi_1}\right)^{e/2}}{\tan\left(\frac{\pi}{4} - \frac{\phi_2}{2}\right)\left(\frac{1 + e\sin\phi_2}{1 - e\sin\phi_2}\right)^{e/2}} \right\}^{\sin\phi_0} . \tag{5.3.27}$$

From (5.3.26) and (5.3.27)

$$\frac{R_{p1}\cos\phi_1}{R_{p2}\cos\phi_2} = \left\{ \frac{\tan\left(\frac{\pi}{4} - \frac{\phi_1}{2}\right)\left(\frac{1 + e\sin\phi_1}{1 - e\sin\phi_1}\right)^{e/2}}{\tan\left(\frac{\pi}{4} - \frac{\phi_2}{2}\right)\left(\frac{1 + e\sin\phi_2}{1 - e\sin\phi_2}\right)^{e/2}} \right\}^{\sin\phi_0}$$

$$\ln\left(\frac{R_{p1}\cos\phi_1}{R_{p2}\cos\phi_2}\right) = \sin\phi_0 \ln \left\{ \frac{\tan\left(\frac{\pi}{4} - \frac{\phi_1}{2}\right)\left(\frac{1 + e\sin\phi_1}{1 - e\sin\phi_1}\right)^{e/2}}{\tan\left(\frac{\pi}{4} - \frac{\phi_2}{2}\right)\left(\frac{1 + e\sin\phi_2}{1 - e\sin\phi_2}\right)^{e/2}} \right\}$$

$$\sin\phi_0 = \frac{\ln\left(\frac{R_{p1}\cos\phi_1}{R_{p2}\cos\phi_2}\right)}{\ln \left\{ \frac{\tan\left(\frac{\pi}{4} - \frac{\phi_1}{2}\right)\left(\frac{1 + e\sin\phi_1}{1 - e\sin\phi_1}\right)^{e/2}}{\tan\left(\frac{\pi}{4} - \frac{\phi_2}{2}\right)\left(\frac{1 + e\sin\phi_2}{1 - e\sin\phi_2}\right)^{e/2}} \right\}} . \tag{5.3.28}$$

So far, the conical surface has been considered to be tangent at the central circle of parallel. In order to require secancy at ϕ_1 and ϕ_2, let $m = 1$ in (5.3.25)

$$\frac{\rho_1 \sin\phi_0}{R_{p1}\cos\phi_1} = \frac{\rho_2\sin\phi_0}{R_{p2}\cos\phi_2} = 1$$

$$\left.\begin{aligned} \rho_1\sin\phi_0 &= R_{p1}\cos\phi_1 \\ \rho_2\sin\phi_0 &= R_{p2}\cos\phi_2 \end{aligned}\right\} . \tag{5.3.29}$$

In (5.3.29), $\sin\phi_0$, as defined in (5.3.28), applies. From (5.3.21)

$$\rho_1 = \rho_0 \left\{ \frac{\tan\left(\frac{\pi}{4} - \frac{\phi_1}{2}\right)\left(\frac{1 + e\sin\phi_1}{1 - e\sin\phi_1}\right)^{e/2}}{\tan\left(\frac{\pi}{4} - \frac{\phi_0}{2}\right)\left(\frac{1 + e\sin\phi_0}{1 - e\sin\phi_0}\right)^{e/2}} \right\}^{\sin\phi_0} . \tag{5.3.30}$$

Substitute (5.3.30) into (5.3.29).

$$\rho_0 \left\{ \frac{\tan\left(\frac{\pi}{4} - \frac{\phi_1}{2}\right)\left(\frac{1 + e\sin\phi_1}{1 - e\sin\phi_1}\right)^{e/2}}{\tan\left(\frac{\pi}{4} - \frac{\phi_0}{2}\right)\left(\frac{1 + e\sin\phi_0}{1 - e\sin\phi_0}\right)^{e/2}} \right\}^{\sin\phi_0} = R_{p1}\cos\phi_1 .$$

Let

$$\psi = \frac{\rho_0}{\tan\left(\frac{\pi}{4} - \frac{\phi_0}{2}\right)\left(\frac{1 + e\sin\phi_0}{1 - e\sin\phi_0}\right)^{e/2}}$$

$$= \frac{R_{p1}\cos\phi_1}{\sin\phi_0 \tan\left(\frac{\pi}{4} - \frac{\phi_1}{2}\right)\left(\frac{1 + e\sin\phi_1}{1 - e\sin\phi_1}\right)^{e/2}} . \tag{5.3.31}$$

In a similar manner

$$\psi = \frac{R_{p2}\cos\phi_2}{\sin\phi_0 \tan\left(\frac{\pi}{4} - \frac{\phi_2}{2}\right)\left(\frac{1 + e\sin\phi_2}{1 - e\sin\phi_2}\right)^{e/2}} .$$

The polar equations become

$$\theta = \lambda \sin\phi_0$$

$$\rho = \psi \left\{ \tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right)\left(\frac{1 + e\sin\phi}{1 - e\sin\phi}\right)^{e/2} \right\} . \tag{5.3.32}$$

Figure 5.3.2 displays a Lambert conformal projection with two standard parallels developed by using (5.3.32) in conjunction with (5.3.1). Table 5.3.2 gives the plotting coordinates for the two standard parallel case. Notice that the meridians are straight lines, and the parallels are concentric circles. The area between the standard parallels is smaller than on the spheroid, and the area beyond the standard parallels is larger.

Equations (5.3.28), (5.3.31) and (5.3.32) can be converted to the transformation from the conformal sphere by setting e = 0.

$$\sin\phi_0 = \frac{\ln\left(\dfrac{\cos\phi_1}{\cos\phi_2}\right)}{\ln\left[\dfrac{\left(\tan\dfrac{\pi}{4} - \dfrac{\phi_1}{2}\right)}{\left(\tan\dfrac{\pi}{4} - \dfrac{\phi_2}{2}\right)}\right]} \tag{5.3.33}$$

$$\psi = \frac{a\cos\phi_1}{\sin\phi_0\left[\tan\left(\dfrac{\pi}{4} - \dfrac{\phi_1}{2}\right)\right]} = \frac{a\cos\phi_2}{\sin\phi_0\left[\tan\left(\dfrac{\pi}{4} - \dfrac{\phi_2}{2}\right)\right]} \tag{5.3.34}$$

$$\rho = \psi\tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right). \tag{5.3.35}$$

Equations (5.3.33), (5.3.34), and (5.3.35) are used with (5.3.1) to obtain the plotting equations.

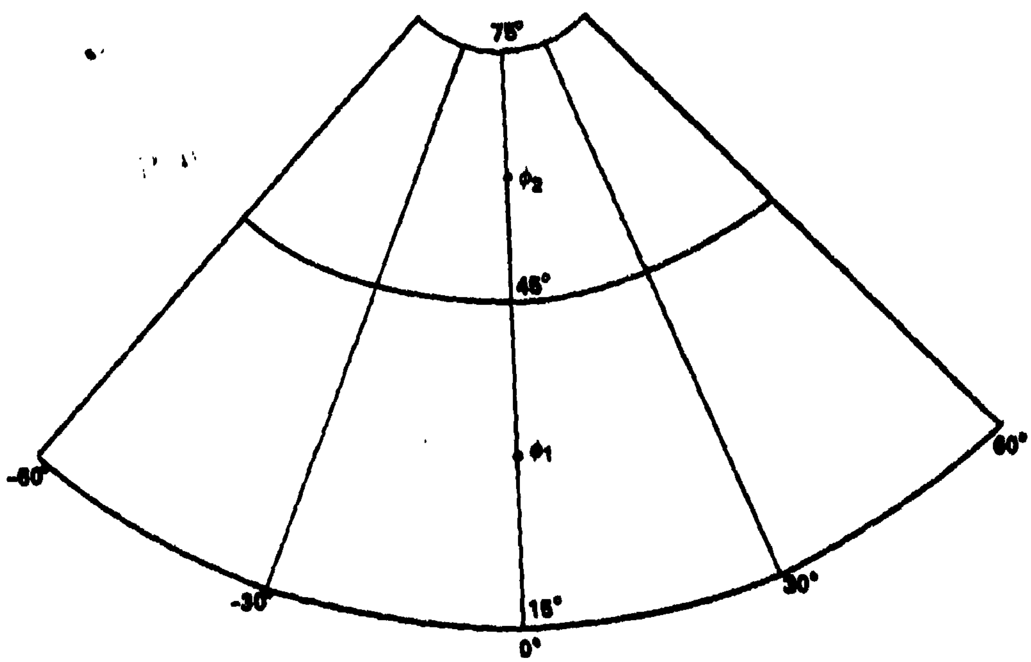


Figure 5.3.2. Lambert conformal projection, two standard parallels

Table 5.3.2. Lambert Conformal Projection, Two Standard Parallels.

Lambert Conformal, Two Standard Parallels

Latitude*	Longitude*	X**	Y**
15.0000	0.0000	0.000	-2.525
15.0000	15.0000	1.909	-2.346
15.0000	30.0000	3.752	-1.814
15.0000	45.0000	5.463	-.948
15.0000	60.0000	6.983	.221
30.0000	0.0000	0.000	0.000
30.0000	15.0000	1.439	.135
30.0000	30.0000	2.827	.536
30.0000	45.0000	4.117	1.169
30.0000	60.0000	5.262	2.070
45.0000	0.0000	0.000	2.175
45.0000	15.0000	1.034	2.272
45.0000	30.0000	2.031	2.560
45.0000	45.0000	2.959	3.029
45.0000	60.0000	3.781	3.662
60.0000	0.0000	0.000	4.131
60.0000	15.0000	.669	4.194
60.0000	30.0000	1.315	4.380
60.0000	45.0000	1.915	4.684
60.0000	60.0000	2.448	5.094
75.0000	0.0000	0.000	5.958
75.0000	15.0000	.329	5.989
75.0000	30.0000	.647	6.080
75.0000	45.0000	.942	6.230
75.0000	60.0000	1.204	6.431

$\phi_1 = 30°$ *Degrees

$\phi_2 = 60°$ **Meters

$\lambda_0 = 0°$

5.4 Stereographic Projections [20], [22], [23], [24]

The stereographic projections entail the transformation from the spheroid to the plane. Three variations of the stereographic projection will be derived. These are the polar, the oblique, and the equatorial.

The stereographic projection may be considered as a purely geometrical projection. This is illustrated best with the projection from the conformal sphere to the plane tangent at the pole. The geometry of this projection is given in Figure 5.4.1.

The plane is tangent to the sphere at the north pole, N. The rays emanate from the south pole, S. The principle of the stereographic projection requires that the projection point be diametrically across from the point of tangency. A typical ray from S to a point P on the earth is transformed to the position P′ on the plane. Thus, the entire projection can be derived by elementary trigonometry. The same is true for the spheroidal case, only the geometry is considerably more messy.

The approach in this section is mathematical rather than geometrical. This approach brings out the quality of conformality immediately. We will consider the plane as one of the limiting forms of the cone, and then apply the equations already derived for the Lambert conformal projection with one standard parallel. This is done by letting the parallel of tangency for the spheroidal case shrink to a polar point of tangency in order to derive the polar stereographic projection. To this end, let $\phi_0 = 90°$ in (5.3.19), (5.3.22), (5.3.14), and (5.3.21).

$$\sin \phi_0 = 1 \tag{5.4.1}$$

$$\theta = \lambda \tag{5.4.2}$$

$$\begin{aligned} \rho_0 &= R_{p0} \cot \phi_0 \\ &= \frac{a}{\sqrt{1 - e^2}} \cot \phi_0 \end{aligned} \tag{5.4.3}$$

and

$$\begin{aligned} \rho &= \frac{a \cot \phi_0}{\sqrt{1 - e^2}} \left\{ \frac{\tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right)\left(\frac{1 + e\sin\phi}{1 - e\sin\phi}\right)^{e/2}}{\tan\left(\frac{\pi}{4} - \frac{\phi_0}{2}\right)\left(\frac{1 + e}{1 - e}\right)^{e/2}} \right\} \\ &= \tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right)\left(\frac{1 + e\sin\phi}{1 - e\sin\phi}\right)^{e/2} \frac{a}{\sqrt{1 - e^2}} \left(\frac{1 - e}{1 + e}\right)^{e/2} \\ &\quad \times \frac{\tan\left(\frac{\pi}{4} + \frac{\phi_0}{2}\right)}{\tan \phi_0} . \end{aligned} \tag{5.4.4}$$

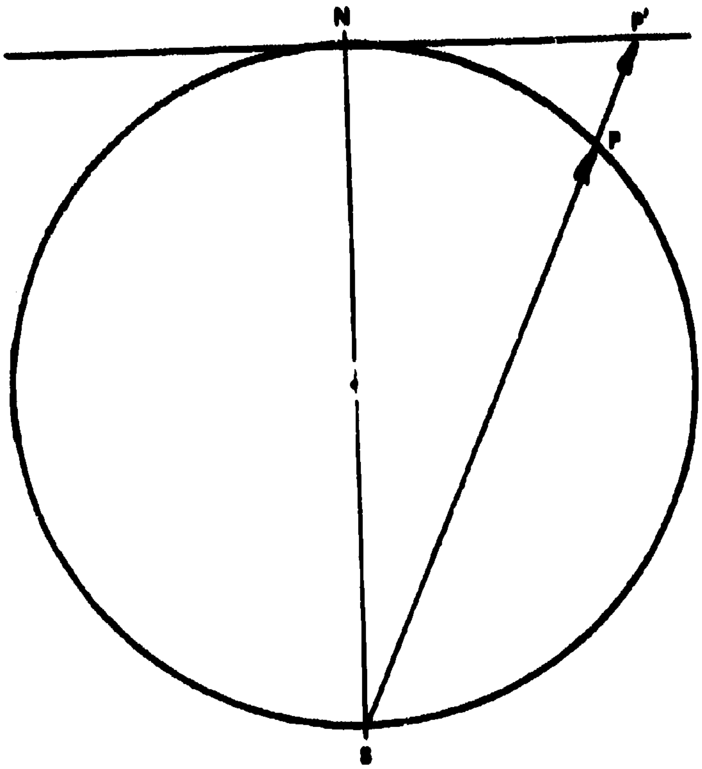


Figure 5.4.1. Geometry for the stereographic projection

In (5.4.4), take the limit of

$$\frac{\tan\left(\frac{\pi}{4} + \frac{\phi_0}{2}\right)}{\tan \phi_0},$$

as ϕ_0 approaches 90°.

$$\underset{\phi_0 \to 90^\circ}{\text{L}} \frac{\tan\left(\frac{\pi}{4} + \frac{\phi_0}{2}\right)}{\tan \phi_0} = \underset{\phi_0 \to 90^\circ}{\text{L}} \left[\left(\frac{1 + \tan \phi_0/2}{1 - \tan \phi_0/2}\right)\left(\frac{1 - \tan^2 \phi_0/2}{2 \tan \phi_0/2}\right)\right]$$

$$= \underset{\phi_0 \to 90^\circ}{\text{L}} \left[\frac{(1 + \tan \phi_0/2)^2}{2 \tan \phi_0/2}\right] = 2\,. \qquad (5.4.5)$$

Substitute (5.4.5) into (5.4.4).

$$\rho = \frac{2a}{\sqrt{1 - e^2}} \left(\frac{1 - e}{1 + e}\right)^{e/2} \tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right)\left(\frac{1 + e \sin \phi}{1 - e \sin \phi}\right)^{e/2}. \qquad (5.4.6)$$

The Cartesian plotting coordinates for the polar stereographic projection are

$$\left.\begin{aligned} x &= \rho \sin \theta \\ y &= -\rho \cos \theta \end{aligned}\right\}. \qquad (5.4.7)$$

Equations (5.4.7) are evaluated using (5.4.3) and (5.4.6).

Equation (5.4.7) can be converted into a transformation from the conformal sphere to the plane by letting $e = 0$. Then

$$\rho = 2a \tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right). \qquad (5.4.8)$$

Substitute (5.4.3) and (5.4.8) into (5.4.7).

$$\left.\begin{aligned} x &= 2a \tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right) \sin \lambda \\ y &= -2a \tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right) \cos \lambda \end{aligned}\right. \qquad (5.4.9)$$

Figure 5.4.2 gives an example of the polar stereographic projection. Note that the meridians are straight lines converging on the pole, and the parallels are concentric circles centered on the pole. The spacing between the parallels increases as one goes towards the equator. Table 5.4.1 gives the plotting coordinates.

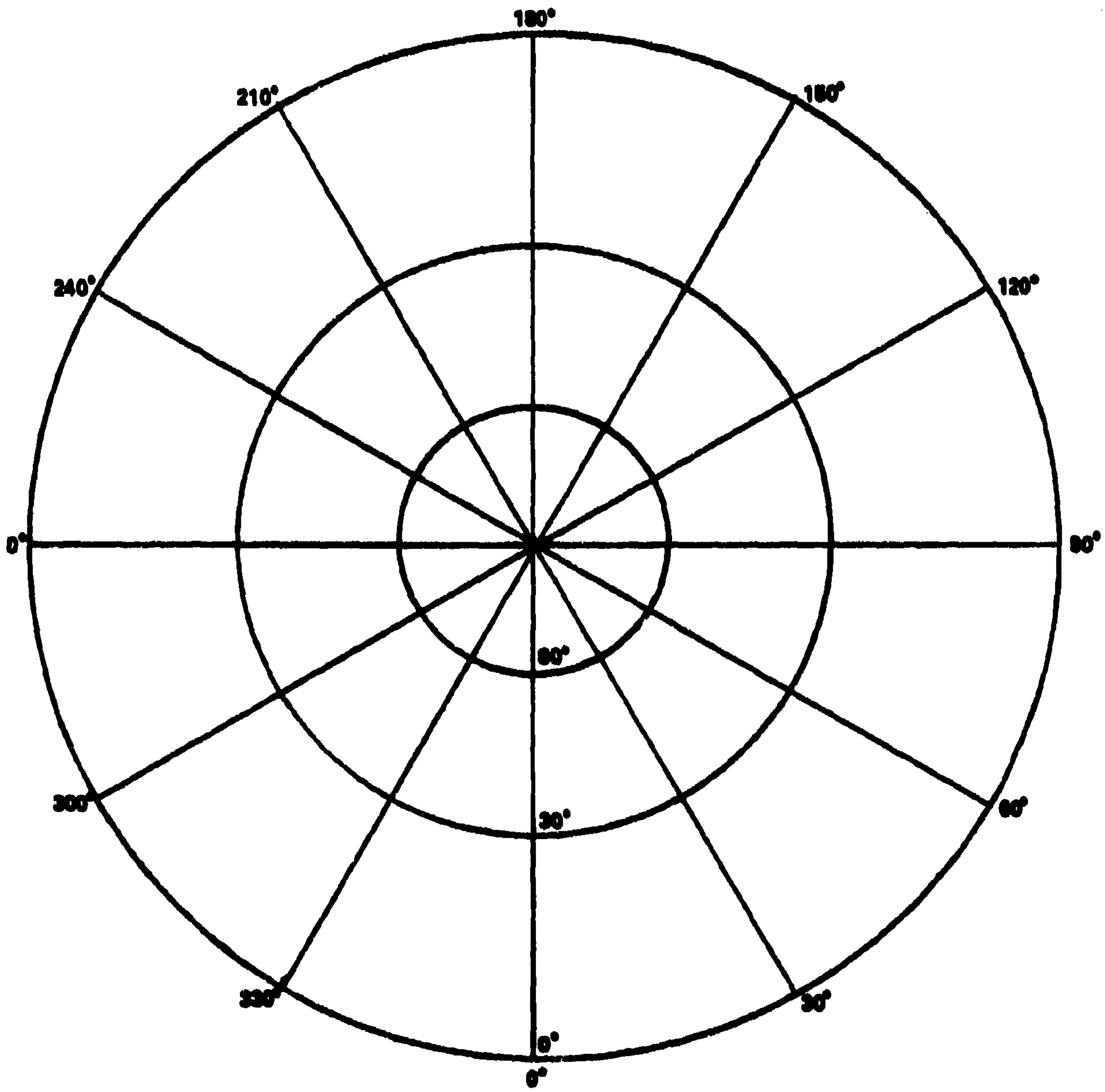


Figure 5.4.2. Polar stereographic projection

Table 5.4.1. Stereographic Projection, Polar Case.

Stereographic Projection, Polar Case

Latitude*	Longitude*	X**	Y**
0.0000	0.0000	12.714	0.000
0.0000	15.0000	12.280	3.291
0.0000	30.0000	11.010	6.357
0.0000	45.0000	8.990	8.990
0.0000	60.0000	6.357	11.010
0.0000	75.0000	3.290	12.280
0.0000	90.0000	-.000	12.714
15.0000	0.0000	9.772	0.000
15.0000	15.0000	9.439	2.529
15.0000	30.0000	8.463	4.886
15.0000	45.0000	6.910	6.910
15.0000	60.0000	4.886	8.463
15.0000	75.0000	2.529	9.439
15.0000	90.0000	-.000	9.772
30.0000	0.0000	7.365	0.000
30.0000	15.0000	7.114	1.906
30.0000	30.0000	6.378	3.682
30.0000	45.0000	5.208	5.208
30.0000	60.0000	3.682	6.378
30.0000	75.0000	1.906	7.114
30.0000	90.0000	-.000	7.365
45.0000	0.0000	5.291	0.000
45.0000	15.0000	5.111	1.369
45.0000	30.0000	4.582	2.646
45.0000	45.0000	3.741	3.741
45.0000	60.0000	2.645	4.582
45.0000	75.0000	1.369	5.111
45.0000	90.0000	-.000	5.291
60.0000	0.0000	3.426	0.000
60.0000	15.0000	3.310	.887
60.0000	30.0000	2.967	1.713
60.0000	45.0000	2.423	2.423
60.0000	60.0000	1.713	2.967
60.0000	75.0000	.887	3.310
60.0000	90.0000	-.000	3.426
75.0000	0.0000	1.685	0.000
75.0000	15.0000	1.627	.436
75.0000	30.0000	1.459	.842
75.0000	45.0000	1.191	1.191
75.0000	60.0000	.842	1.459
75.0000	75.0000	.436	1.627
75.0000	90.0000	-.000	1.685
90.0000	0.0000	-.000	0.000
90.0000	15.0000	-.000	-.000
90.0000	30.0000	-.000	-.000
90.0000	45.0000	-.000	-.000
90.0000	60.0000	-.000	-.000
90.0000	75.0000	-.000	-.000
90.0000	90.0000	.000	-.000

$\phi_0 = 90°$ $\lambda_0 = 0°$ *Degrees **Meters

The oblique case for the stereographic transformation from the conformal sphere to the plane may be obtained by applying the transformation formulas of Section 2.10. The latitude and longitude of the pole of the auxiliary coordinate system is ϕ_p and λ_p, respectively. Write (5.4.9) as

$$\left.\begin{aligned} x &= 2a \tan\left(\frac{\pi}{4} - \frac{h}{2}\right) \sin\alpha \\ y &= -2a \tan\left(\frac{\pi}{4} - \frac{h}{2}\right) \cos\alpha \end{aligned}\right\}. \tag{5.4.10}$$

Then, from Section 2.10,

$$\alpha = \tan \left\{\frac{\sin(\lambda - \lambda_p)}{\cos\phi_p \tan\phi - \sin\phi_p \cos(\lambda - \lambda_p)}\right\} \tag{5.4.11}$$

$$h = \sin^{-1}\{\sin\phi \sin\phi_p + \cos\phi \cos\phi_p \cos(\lambda - \lambda_p)\}\,. \tag{5.4.12}$$

Equations (5.4.10), (5.4.11), and (5.4.12) will then produce a grid such as the one in Figure 5.4.3. The only straight line in this projection is the central meridian. Table 5.4.2 is the plotting table for $\phi_p = 45°$.

The equatorial case follows when $\phi_p = 0°$ in (5.4.11) and (5.4.12). These equations simplify to give

$$\left.\begin{aligned} \alpha &= \tan^{-1}\left\{\frac{\sin(\lambda - \lambda_p)}{\tan\phi}\right\} \\ h &= \sin^{-1}\{\cos\phi \cos(\lambda - \lambda_p)\} \end{aligned}\right\}. \tag{5.4.13}$$

Figure 5.4.4 shows an equatorial stereographic projection. The equator and central meridian are the only straight lines on the grid. The other lines are arcs of ellipses. The plotting coordinates are in Table 5.4.3.

The stereographic projection can also be derived for a transformation from the conformal sphere by a process similar to that introduced for the gnomonic, azimuthal equidistant and orthographic projections of Chapter 6.

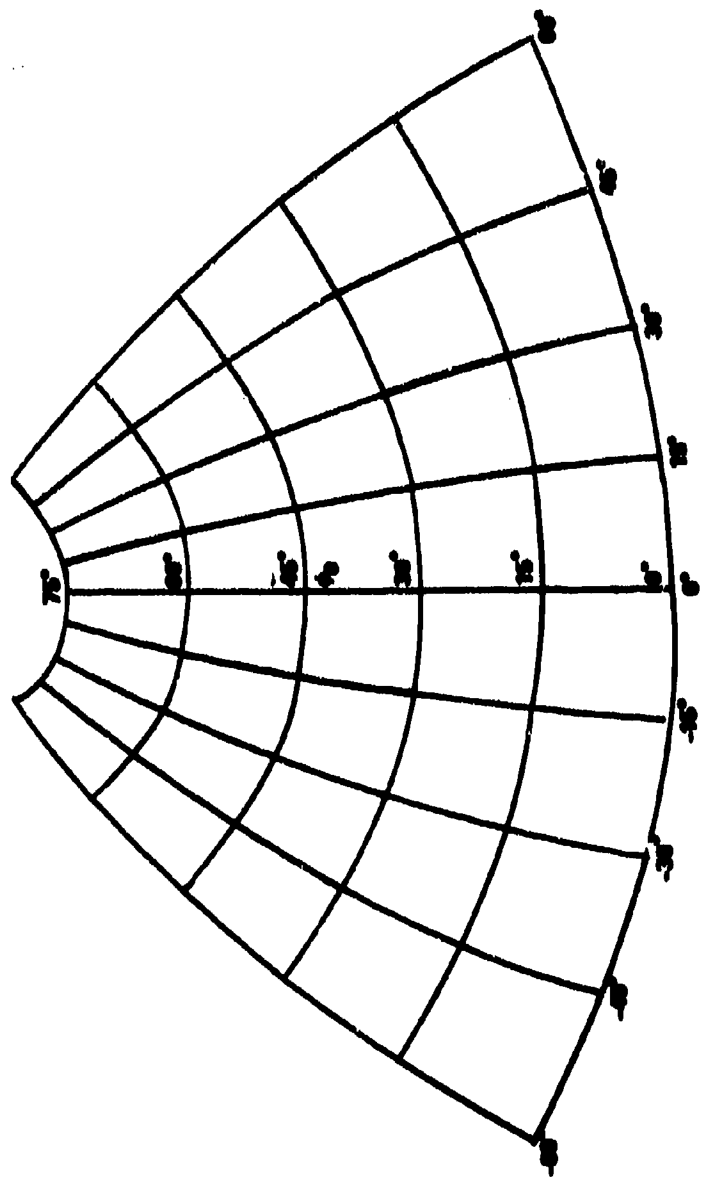


Figure 5.4.3. Oblique stereographic projection

Table 8.4.2. Stereographic Projection, Oblique Case.

Stereographic Projection, Oblique Case

Latitude*	Longitude*	X**	Y**
0.0000	0.0000	0.000	-5.284
0.0000	15.0000	1.962	-5.177
0.0000	30.0000	3.956	-4.845
0.0000	45.0000	6.013	-4.252
0.0000	60.0000	8.162	-3.332
0.0000	75.0000	10.416	-1.973
0.0000	90.0000	12.756	.000
15.0000	0.0000	0.000	-3.418
15.0000	15.0000	1.731	-3.300
15.0000	30.0000	3.472	-2.977
15.0000	45.0000	5.230	-2.297
15.0000	60.0000	7.000	-1.326
15.0000	75.0000	8.753	.059
15.0000	90.0000	10.416	1.974
30.0000	0.0000	0.000	-1.679
30.0000	15.0000	1.470	-1.601
30.0000	30.0000	2.932	-1.197
30.0000	45.0000	4.372	-.567
30.0000	60.0000	5.764	.364
30.0000	75.0000	7.057	1.646
30.0000	90.0000	8.162	3.332
45.0000	0.0000	0.000	.000
45.0000	15.0000	1.177	.110
45.0000	30.0000	2.333	.442
45.0000	45.0000	3.441	1.008
45.0000	60.0000	4.464	1.922
45.0000	75.0000	5.347	2.901
45.0000	90.0000	6.013	4.252
60.0000	0.0000	0.000	1.679
60.0000	15.0000	.845	1.768
60.0000	30.0000	1.662	2.036
60.0000	45.0000	2.422	2.482
60.0000	60.0000	3.087	3.106
60.0000	75.0000	3.616	3.908
60.0000	90.0000	3.956	4.845
75.0000	0.0000	0.000	3.418
75.0000	15.0000	.459	3.472
75.0000	30.0000	.896	3.633
75.0000	45.0000	1.288	3.896
75.0000	60.0000	1.611	4.252
75.0000	75.0000	1.843	4.686
75.0000	90.0000	1.962	5.177
90.0000	0.0000	0.000	5.284
90.0000	15.0000	-.000	5.284
90.0000	30.0000	-.000	5.284
90.0000	45.0000	-.000	5.284
90.0000	60.0000	-.000	5.284
90.0000	75.0000	-.000	5.284
90.0000	90.0000	-.000	5.284

$\phi_0 = 45°$ $\lambda_0 = 0°$ *Degrees **Meters

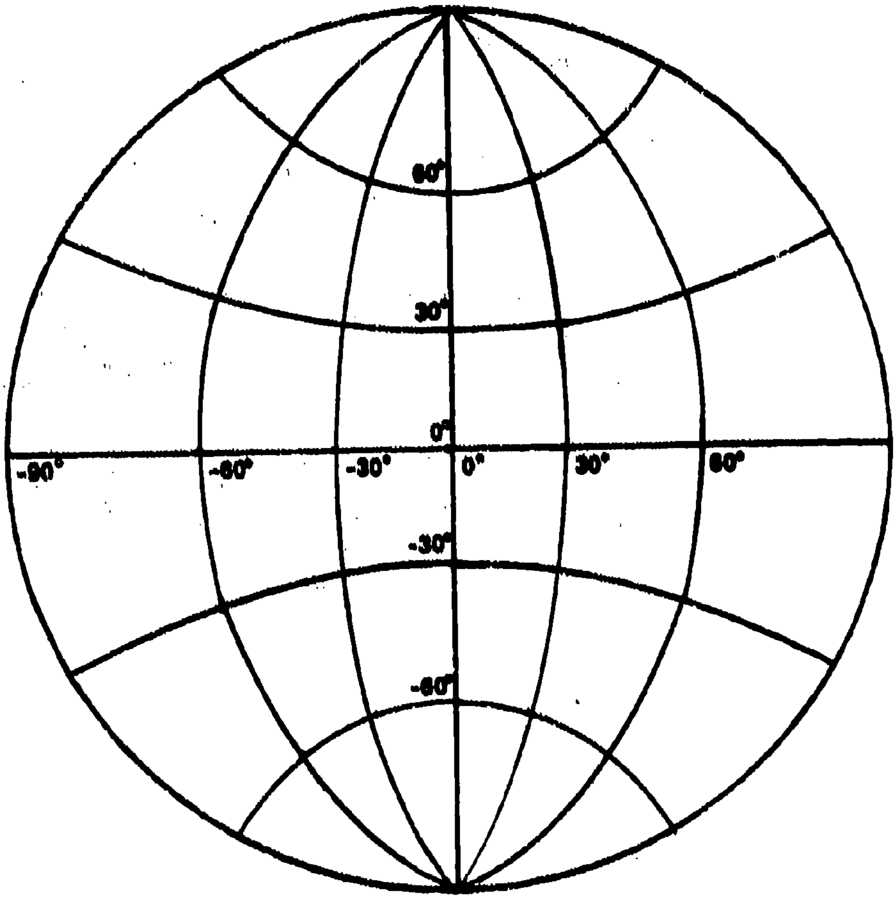


Figure 5.4.4. Stereographic projection, equatorial case

Table 8.4.3. Stereographic Projection, Equatorial Case.

Stereographic Projection, Equatorial Case

Latitude*	Longitude*	X**	Y**
0.0000	0.0000	0.000	0.000
0.0000	15.0000	1.679	0.000
0.0000	30.0000	3.418	0.000
0.0000	45.0000	5.284	0.000
0.0000	60.0000	7.365	0.000
0.0000	75.0000	9.768	0.000
0.0000	90.0000	12.757	0.000
15.0000	0.0000	0.000	1.679
15.0000	15.0000	1.650	1.708
15.0000	30.0000	3.355	1.798
15.0000	45.0000	5.177	1.962
15.0000	60.0000	7.196	2.226
15.0000	75.0000	9.322	2.641
15.0000	90.0000	12.322	3.302
30.0000	0.0000	0.000	3.418
30.0000	15.0000	1.557	3.473
30.0000	30.0000	3.156	3.645
30.0000	45.0000	4.845	3.956
30.0000	60.0000	6.676	4.451
30.0000	75.0000	8.717	5.210
30.0000	90.0000	11.047	6.378
45.0000	0.0000	0.000	5.284
45.0000	15.0000	1.387	5.360
45.0000	30.0000	2.797	5.594
45.0000	45.0000	4.252	6.013
45.0000	60.0000	5.771	6.664
45.0000	75.0000	7.365	7.675
45.0000	90.0000	9.020	9.020
60.0000	0.0000	0.000	7.365
60.0000	15.0000	1.113	7.470
60.0000	30.0000	2.225	7.709
60.0000	45.0000	3.332	8.162
60.0000	60.0000	4.419	8.838
60.0000	75.0000	5.455	9.782
60.0000	90.0000	6.378	11.047
75.0000	0.0000	0.000	9.788
75.0000	15.0000	.684	9.857
75.0000	30.0000	1.348	10.056
75.0000	45.0000	1.973	10.416
75.0000	60.0000	2.532	10.910
75.0000	75.0000	2.989	11.548
75.0000	90.0000	3.301	12.322
90.0000	0.0000	0.000	12.757
90.0000	15.0000	-.000	12.757
90.0000	30.0000	-.000	12.757
90.0000	45.0000	-.000	12.756
90.0000	60.0000	-.000	12.756
90.0000	75.0000	-.000	12.756
90.0000	90.0000	-.000	12.756

$\phi_0 = 0^\circ$ $\lambda_0 = 0^\circ$ *Degrees **Meters

Chapter 6

CONVENTIONAL PROJECTIONS

Conventional projections are those which are neither equal area nor conformal. As was mentioned in Chapter 1, this is not a derogatory term. The conventional projections were produced in order to preserve some special quality which is more important to a particular cartographer than equal area or conformality, or to present a projection which is either mathematically or graphically simple. Since the category of conventional is a catch-all, it is to be expected that there is a wide variety in this class of projections. This is true. Some of these are really of historical interest. Some are simply convenient. Others have proved to be cartographic work-horses.

The most useful of the conventional projections are the gnomonic, the azimuthal equidistant, and the polyconic, both regular and transverse. The simple geometrical projections of the conical and cylindrical, as well as the orthographic projection of the engineer, are examples of strictly geometrical approaches to the problem. Of mainly historical interest, are the Van der Grinten, the plate carrée, the carte parallelogrammatique, the Gall, the Murdoch, the Cassini, and the stereographic variations such as the Clarke, the James, and the La Hire. Finally, there are such mathematical endeavours as the globular.

Clearly, the conventional projections provide maps ranging from the most utilitarian to the unique.

6.1 Gnomonic Projection [8], [20]

The gnomonic projection requires that the transformation of positions on the surface of the earth onto a plane be based upon a projection point at the center of the earth. The name comes from a gnome's-eye view of the world. The gnomonic projection can be a purely geometrical construction. However, we shall use spherical trigonometry to obtain the oblique gnomonic projection. Then, the two limiting cases of the projection, the polar and the equatorial, will be obtained by particularizing the oblique case.

Figure 6.1.1 portrays the geometry required for deriving the oblique gnomonic projection. Let a plane be tangent to the sphere at point O, whose coordinates are (ϕ_0, λ_0). The Cartesian axes on the plane are such that x is east, and y is north. Let an arbitrary point P, with coordinates (ϕ, λ) be projected onto the plane to become P′, with mapping coordinates (x, y).

Define an auxiliary angle, ψ, between the radius vectors CO and CP. From the figure

$$OP' = a \tan \psi \tag{6.1.1}$$

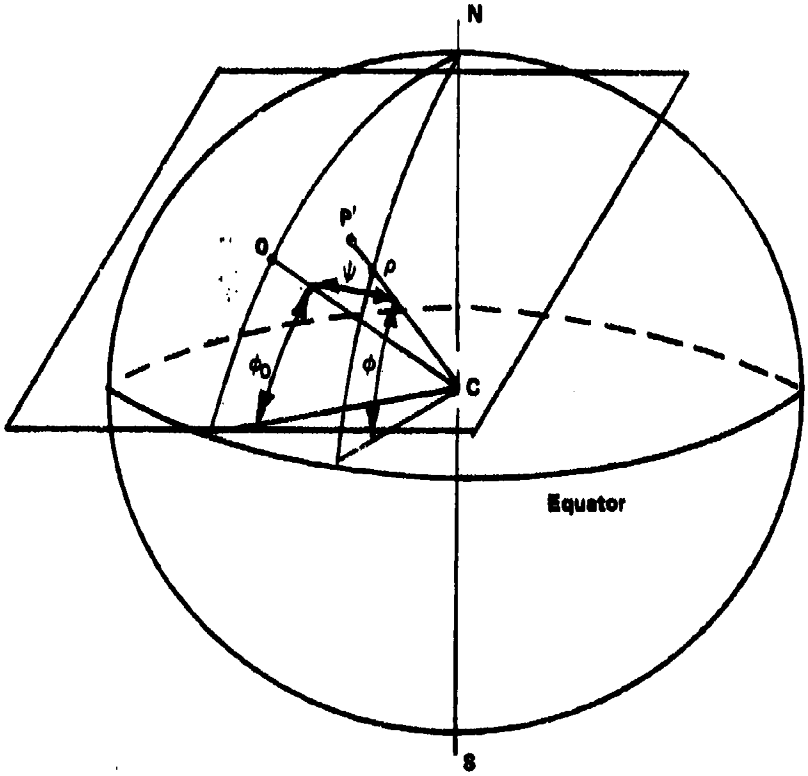


Figure 6.1.1. Geometry for the oblique gnomonic projection

On the mapping plane, define the second auxiliary angle, θ, which orients OP′ with respect to the x-axis.

$$x = OP' \cos\theta \tag{6.1.2}$$

Substitute (6.1.1) into (6.1.2).

$$x = a \tan\psi \cos\theta$$

$$= \frac{a \sin\psi}{\cos\psi} \cos\theta \tag{6.1.3}$$

Also,

$$y = OP' \sin\theta \tag{6.1.4}$$

Substitute (6.1.1) into (6.1.4).

$$y = a \tan\psi \sin\theta$$

$$= \frac{a \sin\psi}{\cos\psi} \sin\theta \tag{6.1.5}$$

It is now necessary to find ψ and θ in terms of ϕ, ϕ_0, λ, and λ_0. From Figure 6.1.1, by the use of the law of sines

$$\frac{\sin(\lambda - \lambda_0)}{\sin\psi} = \frac{\sin(90^\circ - \theta)}{\sin(90^\circ - \phi)}$$

$$= \frac{\cos\theta}{\cos\phi}$$

$$\sin\psi \cos\theta = \sin(\lambda - \lambda_0) \cos\phi \tag{6.1.6}$$

Apply the law of cosines.

$$\cos\psi = \cos(90^\circ - \phi_0) \cos(90^\circ - \phi)$$

$$+ \sin(90^\circ - \phi_0) \sin(90^\circ - \phi) \cos(\lambda - \lambda_0)$$

$$= \sin\phi_0 \sin\phi + \cos\phi_0 \cos\phi \cos(\lambda - \lambda_0) \tag{6.1.7}$$

Apply equation (2.10.6)

$$\sin\psi \cos(90^\circ - \theta) = \sin(90^\circ - \phi_0) \cos(90^\circ - \phi)$$

$$- \cos(90^\circ - \phi_0) \sin(90^\circ - \phi) \cos(\lambda - \lambda_0)$$

$$\sin\psi \sin\theta = \cos\phi_0 \sin\phi$$

$$- \sin\phi_0 \cos\phi \cos(\lambda - \lambda_0) \tag{6.1.8}$$

Substitute (6.1.6), and (6.1.7) into (6.1.3), and (6.1.7) and (6.1.8) into (6.1.5).

$$\left.\begin{aligned} x &= \frac{aS \cos\phi \sin(\lambda - \lambda_0)}{\sin\phi_0 \sin\phi + \cos\phi_0 \cos\phi \cos(\lambda - \lambda_0)} \\ y &= \frac{aS\,[\cos\phi_0 \sin\phi - \sin\phi_0 \cos\phi \cos(\lambda - \lambda_0)]}{\sin\phi_0 \sin\phi + \cos\phi_0 \cos\phi \cos(\lambda - \lambda_0)} \end{aligned}\right\} \tag{6.1.9}$$

Equations (6.1.9) are the plotting equations for the oblique gnomonic projection. The grid resulting from a selection of $\phi_0 = 45°$, and $\lambda_0 = 0°$ is given as Figure 6.1.2. In this projection, all the meridians, and the equator are straight lines, since they are great circles. An arbitrary great circle distance between points A and C, on the figure, is also a straight line. The loxodrome between these same two points appears as a curved line. Compare Figure 6.1.2 to Figure 5.1.2 for the equatorial Mercator, in which the situation is reversed. Table 6.1-1 has the plotting coordinates.

To find the gnomonic polar projection, it is necessary to let $\phi_0 = 90°$ in (6.1.9).

$$x = -\frac{aS\cos\phi \sin(\lambda - \lambda_0)}{\sin\phi}$$

$$= -aS \cot\phi \sin(\lambda - \lambda_0) \tag{6.1.10}$$

$$y = \frac{aS \cos\phi \cos(\lambda - \lambda_0)}{\sin\phi}$$

$$= aS \cot\phi \cos(\lambda - \lambda_0) \tag{6.1.11}$$

A polar gnomonic grid is given in Figure 6.1.3, and based on (6.1.10) and (6.1.11). In this case, all meridians again are straight lines. The parallels are concentric circles, whose spacing increases as the latitude decreases. Thus, the distortion becomes extreme as the equator is approached. The equator itself can never be portrayed on the gnomonic polar projection, since a ray from the center of the earth to any point on the equator will be parallel to the projection plane. Table 6.1.2 is the plotting table for the polar projection.

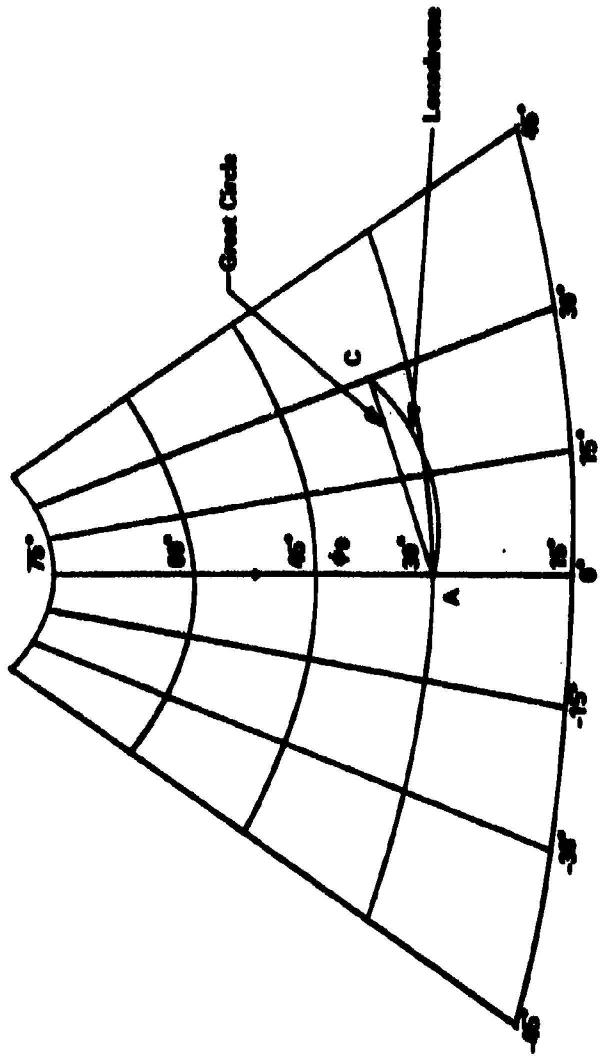


Figure 6.1.2. Oblique gnomonic projection

Table 6.1.1. Gnomonic Projection, Oblique Case.

Gnomonic Oblique Projection

Latitude*	Longitude*	X**	Y**
15.0000	0.0000	0.000	-3.682
15.0000	15.0000	1.892	-3.608
15.0000	30.0000	3.977	-3.364
15.0000	45.0000	6.541	-2.873
30.0000	0.0000	0.000	-1.709
30.0000	15.0000	1.513	-1.606
30.0000	30.0000	3.125	-1.276
30.0000	45.0000	4.966	-.644
45.0000	0.0000	0.000	.000
45.0000	15.0000	1.188	.111
45.0000	30.0000	2.417	.458
45.0000	45.0000	3.736	1.094
60.0000	0.0000	0.000	1.709
60.0000	15.0000	.865	1.811
60.0000	30.0000	1.736	2.126
60.0000	45.0000	2.615	2.680
75.0000	0.0000	0.000	3.682
75.0000	15.0000	.497	3.755
75.0000	30.0000	.981	3.976
75.0000	45.0000	1.437	4.346

$\phi_0 = 45°$ *Degrees

$\lambda_0 = 0°$ **Meters

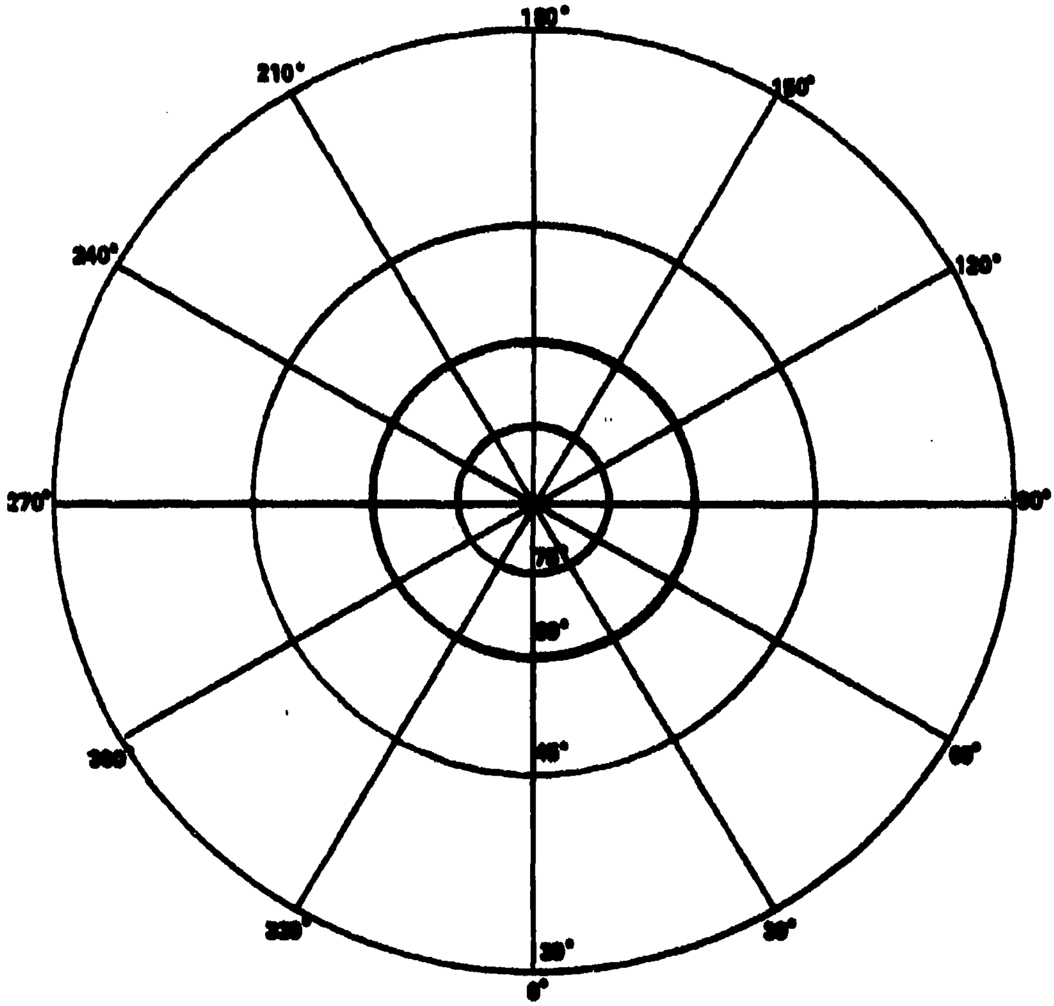


Figure 6.1.3. Polar gnomonic projection

Table 6.1.2. Gnomonic Projection, Polar Case.

Gnomonic Polar Projection

Latitude*	Longitude*	X**	Y**
30.0000	0.0000	0.000	-11.048
30.0000	15.0000	2.859	-10.671
30.0000	30.0000	5.524	-9.567
30.0000	45.0000	7.812	-7.812
30.0000	60.0000	9.567	-5.524
30.0000	75.0000	10.671	-2.859
30.0000	90.0000	11.047	.000
45.0000	0.0000	0.000	-6.378
45.0000	15.0000	1.651	-6.161
45.0000	30.0000	3.189	-5.524
45.0000	45.0000	4.510	-4.510
45.0000	60.0000	5.524	-3.189
45.0000	75.0000	6.161	-1.651
45.0000	90.0000	6.378	-.000
60.0000	0.0000	0.000	-3.682
60.0000	15.0000	.953	-3.557
60.0000	30.0000	1.841	-3.189
60.0000	45.0000	2.604	-2.604
60.0000	60.0000	3.189	-1.841
60.0000	75.0000	3.557	-.953
60.0000	90.0000	3.682	-.000
75.0000	0.0000	0.000	-1.709
75.0000	15.0000	.442	-1.651
75.0000	30.0000	.854	-1.480
75.0000	45.0000	1.208	-1.208
75.0000	60.0000	1.480	-.855
75.0000	75.0000	1.651	-.442
75.0000	90.0000	1.709	-.000
90.0000	0.0000	0.000	-.000
90.0000	15.0000	-.000	-.000
90.0000	30.0000	-.000	-.000
90.0000	45.0000	-.000	-.000
90.0000	60.0000	-.000	-.000
90.0000	75.0000	-.000	-.000
90.0000	90.0000	-.000	-.000

$\phi_0 = 90°$ *Degrees

$\lambda_0 = 0°$ **Meters

The equatorial gnomonic projection is obtained from (6.1.9) by setting $\phi_0 = 0°$.

$$
\left.
\begin{aligned}
x &= -\frac{aS \cos\phi \sin(\lambda - \lambda_0)}{\cos\phi \cos(\lambda - \lambda_0)} \\
&= -aS \tan(\lambda - \lambda_0) \\
y &= \frac{aS \sin\phi}{\cos\phi \cos(\lambda - \lambda_0)}
\end{aligned}
\right\} \qquad (6.1.12)
$$

Figure 6.1.4 gives the equatorial gnomonic projection based on (6.1.12). Again, only the meridians and the equator are straight lines. The plotting coordinates are in Table 6.1.3.

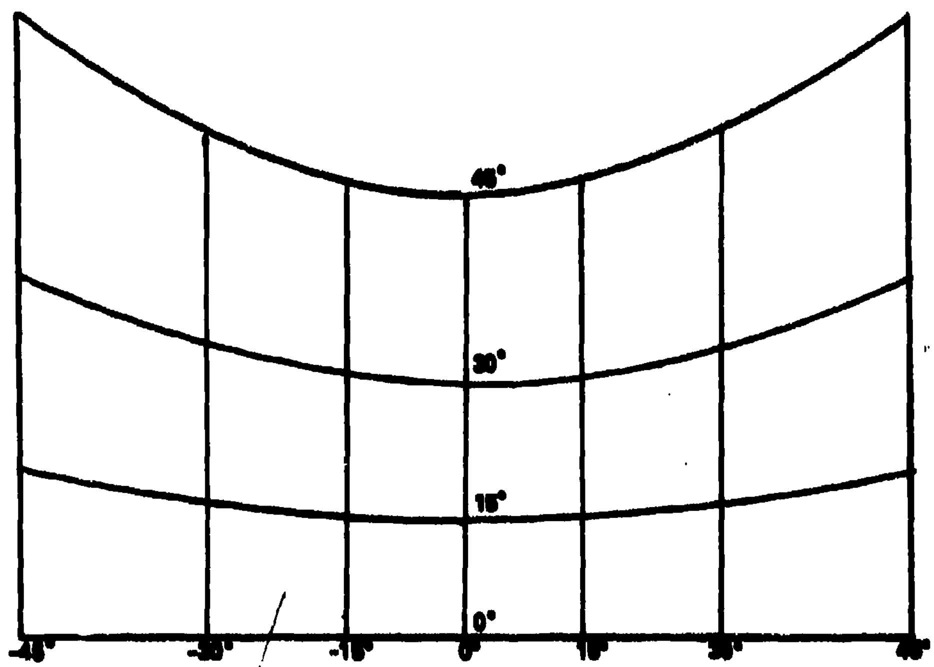


Figure 6.1.4. Gnomonic projection, equatorial case

Table 8.1.3. Gnomonic Projection, Equatorial Case.

Gnomonic Equatorial Projection

Latitude*	Longitude*	X**	Y**
0.0000	0.0000	0.000	0.000
0.0000	15.0000	1.709	0.000
0.0000	30.0000	3.682	0.000
0.0000	45.0000	6.378	0.000
15.0000	0.0000	0.000	1.709
15.0000	15.0000	1.709	1.769
15.0000	30.0000	3.682	1.973
15.0000	45.0000	6.378	2.417
30.0000	0.0000	0.000	3.682
30.0000	15.0000	1.709	3.812
30.0000	30.0000	3.682	4.252
30.0000	45.0000	6.378	5.208
45.0000	0.0000	0.000	6.378
45.0000	15.0000	1.709	6.603
45.0000	30.0000	3.682	7.365
45.0000	45.0000	6.378	9.020

$\phi_0 = 0°$ *Degrees

$\lambda_0 = 0°$ **Meters

6.2 Azimuthal Equidistant Projection [8], [20]

The azimuthal equidistant projection (also called Postel's projection) is another projection directly from the earth onto a plane. The projection law in this case is that the distance and bearing from the origin of the plotting surface to any other point must be true. Thus, all great circles through the origin are lines of true length.

Figure 6.2.1 shows the geometry for the oblique azimuthal equidistant projection. Again, the plane is tangent to the spherical earth at the mapping origin, 0. The coordinates of the origin are (ϕ_0, λ_0). An arbitrary point P on the sphere has coordinates (ϕ, λ). Again, the auxiliary angle between the radius vectors CO and CP is ψ. By the definition of the law of the equidistant transformation

$$OP' = a\psi \tag{6.2.1}$$

$$\left.\begin{aligned} x &= OP' \cos\theta \\ y &= OP' \sin\theta \end{aligned}\right\} \tag{6.2.2}$$

Substitute (6.2.1) into (6.2.2).

$$\left.\begin{aligned} x &= a\psi \cos\theta \\ y &= a\psi \sin\theta \end{aligned}\right\} \tag{6.2.3}$$

The value of ψ is found from (6.1.7).

$$\begin{aligned} \psi = \cos^{-1} \{&\sin\phi_0 \sin\phi \\ &+ \cos\phi_0 \cos\phi \cos(\lambda - \lambda_0)\} \end{aligned} \tag{6.2.4}$$

Since ψ is restricted to the range from 0° to 180°, ψ is uniquely defined. Then, $\sin\psi$ is available immediately. Equations (6.1.6) and (6.1.7) can then be used to obtain θ.

$$\cos\theta = \frac{\sin(\lambda - \lambda_0)\cos\phi}{\sin\psi} \tag{6.2.5}$$

$$\sin\theta = \frac{\cos\phi_0 \sin\phi - \sin\phi_0 \cos\phi \cos(\lambda - \lambda_0)}{\sin\psi} \tag{6.2.6}$$

Equations (6.2.4), (6.2.5), (6.2.6) and (6.2.3), with the introduction of the scale factor, a, are used to produce an oblique azimuthal equidistant grid. Such a grid appears in Figure (6.2.2). Only the central meridian is a straight line. All other meridians, the parallels, and the equator appear as curves of varying degrees of complexity. However, any straight line ruled on the map, from the origin to any arbitrary point will be true length, and true azimuth. The azimuthal equidistant projection has seen much modern use as rocket and missile firing charts, and air route planning charts. Table 6.2.1 gives the plotting coordinates.

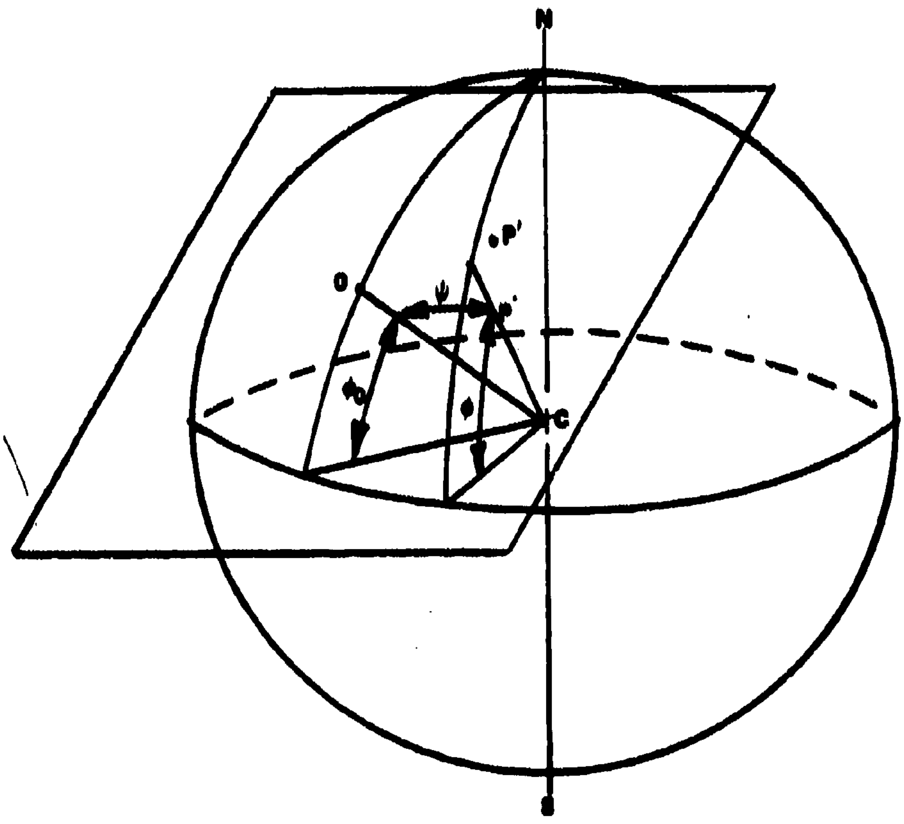


Figure 6.2.1. Geometry for the oblique azimuthal equidistant projection

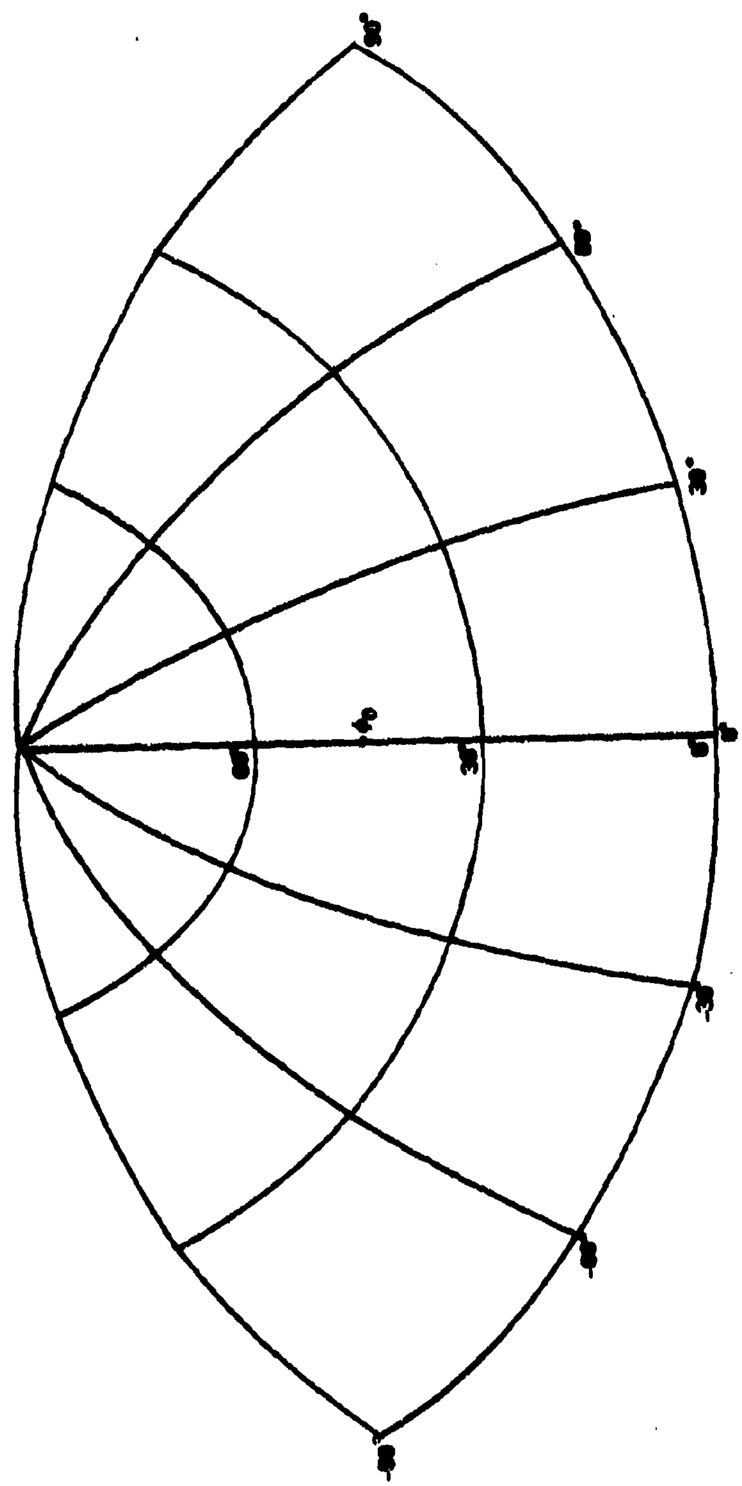


Figure 6.2.2. Oblique azimuthal equidistant projection

Table 6.2.1. Azimuthal Equidistant Projection, Oblique Case.

Azimuthal Equidistant Oblique Projection

Latitude*	Longitude*	X**	Y**
0.0000	0.0000	0.000	-5.009
0.0000	30.0000	3.678	-4.504
0.0000	60.0000	7.142	-2.916
0.0000	90.0000	10.019	.000
30.0000	0.0000	0.000	-1.670
30.0000	30.0000	2.874	-1.173
30.0000	60.0000	5.413	.342
30.0000	90.0000	7.142	2.916
60.0000	0.0000	0.000	1.670
60.0000	30.0000	1.639	2.008
60.0000	60.0000	2.974	2.992
60.0000	90.0000	3.678	4.504
90.0000	0.0000	0.000	5.009
90.0000	30.0000	-.000	5.009
90.0000	60.0000	-.000	5.009
90.0000	90.0000	-.000	5.009

$\phi_0 = 45°$ *Degrees
$\lambda_0 = 0°$ **Meters

In order to obtain the polar azimuthal equidistant projection, let $\phi_0 = 90°$ in (6.2.4), (6.2.5), and (6.2.6). We then have

$$\psi = \cos^{-1} (\sin \phi) \qquad (6.2.7)$$

$$\begin{aligned} \cos \psi &= \sin \phi \\ &= \cos (\pi/2 - \phi) \\ \psi &= \frac{\pi}{2} - \phi \end{aligned} \qquad (6.2.8)$$

Substitute (6.2.8) into (6.2.5) and (6.2.6).

$$\begin{aligned} \cos \theta &= \frac{\sin (\lambda - \lambda_0) \cos \phi}{\sin (\pi/2 - \phi)} \\ &= \sin (\lambda - \lambda_0) \end{aligned} \qquad (6.2.9)$$

$$\begin{aligned} \sin \theta &= - \frac{\cos \phi \cos (\lambda - \lambda_0)}{\sin (\pi/2 - \phi)} \\ &= - \cos (\lambda - \lambda_0) \end{aligned} \qquad (6.2.10)$$

Equations (6.2.3), (6.2.7), (6.2.9) and (6.2.10), with the inclusion of the scale factor, S, give the plotting equations used to develop a grid such as Figure 6.2.3. In this figure, all the meridians are straight lines o. true length, and the parallels are concentric circles, equally spaced. The coordinates for the polar case are in Table 6.2.2.

The equatorial azimuthal equidistant projection is obtained by substituting $\phi_0 = 0°$ into (6.2.4) and (6.2.6).

$$\psi = \cos^{-1} [\cos \phi \cos (\lambda - \lambda_0)] \qquad (6.2.11)$$

$$\sin \theta = \frac{\sin \phi}{\sin \psi} \qquad (6.2.12)$$

The plotting equations are then obtained from (6.2.3), (6.2.5), (6.2.11) and (6.2.12), with the aid of the scale factor, S.

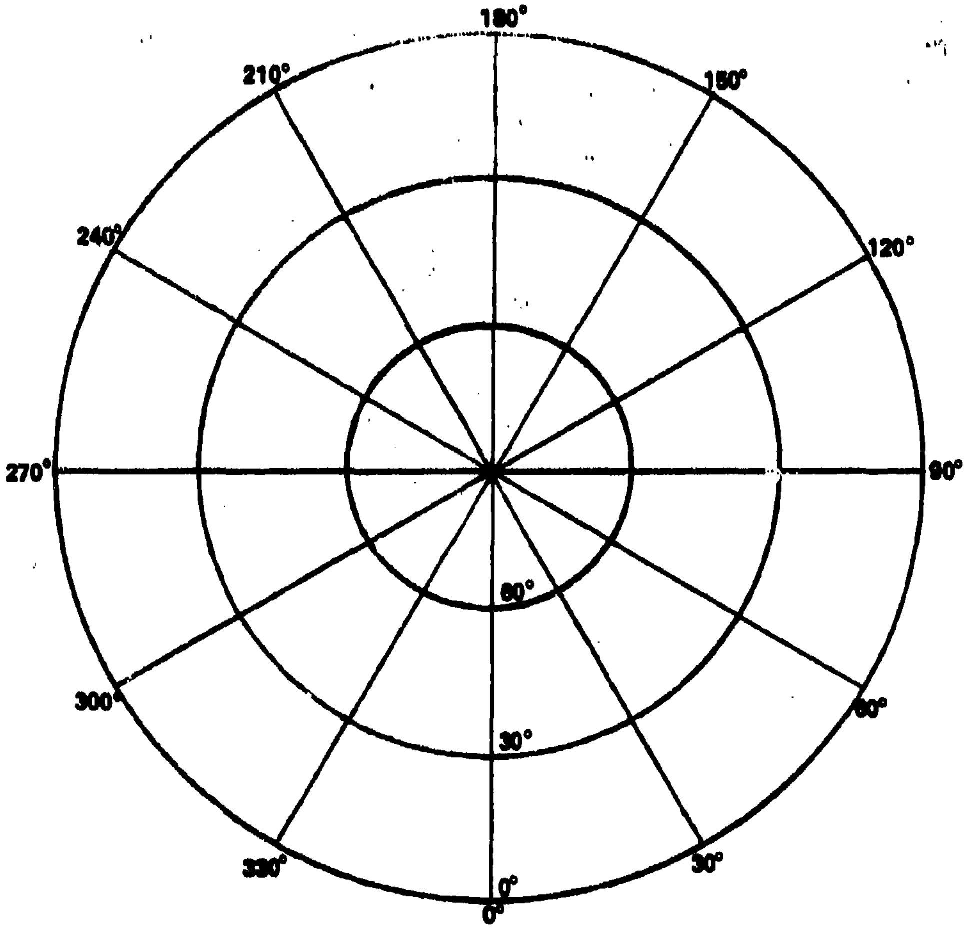


Figure 6.2.3. Polar azimuthal equidistant projection

Table 6.2.2. Azimuthal Equidistant Projection, Polar Case.

Azimuthal Equidistant Polar Projection

Latitude*	Longitude*	X**	Y**
0.0000	0.0000	0.000	-10.019
0.0000	30.0000	5.009	-8.677
0.0000	60.0000	8.677	-5.009
0.0000	90.0000	10.019	.000
30.0000	0.0000	0.000	-6.679
30.0000	30.0000	3.340	-5.784
30.0000	60.0000	5.784	-3.340
30.0000	90.0000	6.679	.000
60.0000	0.0000	0.000	-3.340
60.0000	30.0000	1.670	-2.892
60.0000	60.0000	2.892	-1.670
60.0000	90.0000	3.339	-.000
90.0000	0.0000	0.000	-.000
90.0000	30.0000	-.000	-.000
90.0000	60.0000	-.000	-.000
90.0000	90.0000	-.000	-.000

$\phi_0 = 90°$ *Degrees

$\lambda_0 = 0°$ **Meters

6.3 Orthographic Projection [20], [22]

The orthographic projection is yet another means of portraying the sphere upon the plane by a direct transformation. This is another projection that can be developed by a purely graphical means. In the orthographic projection, the perspective point is placed at infinity. The projection rays fall perpendicularly upon the tangent mapping plane, after intersecting the sphere. The geometry of this projection is shown in Figure 6.3.1 for the oblique case. Only a hemisphere or less can be portrayed on this projection.

Again, the auxiliary angle, ψ, between CO and CP, and the auxiliary angle, θ, on the mapping plane are needed. From the figure

$$OP' = as \sin \psi \tag{6.3.1}$$

$$\left.\begin{array}{l} x = OP' \cos \theta \\ y = OP' \sin \theta \end{array}\right\} \tag{6.3.2}$$

Substitute (6.3.1) into (6.3.2)

$$\left.\begin{array}{l} x = aS \sin \psi \cos \theta \\ y = aS \sin \psi \sin \theta \end{array}\right\} \tag{6.3.3}$$

where s is the scale factor.

Equation (6.1.7) again gives

$$\psi = \cos^{-1} \{\sin \phi_0 \sin \phi + \cos \phi_0 \cos \phi \cos (\lambda - \lambda_0)\} \tag{6.3.4}$$

with sin ψ readily available, since $0 \leq \psi \leq 90$. From (6.1.6) and (6.1.8)

$$\theta = \tan^{-1} \left[\frac{\cos \phi_0 \sin \phi - \sin \phi_0 \cos \phi \cos (\lambda - \lambda_0)}{\sin (\lambda - \lambda_0) \cos \phi}\right] \tag{6.3.5}$$

Equations (6.3.3), (6.3.4), and (6.3.5) give the oblique orthographic projection.

The polar orthographic projection can be obtained from the oblique projection by letting $\phi_0 = 90°$ in (6.3.4) and (6.3.5).

$$\psi = \cos^{-1} (\sin \phi) \tag{6.3.6}$$

$$\theta = \tan^{-1} \left[\frac{-\cos (\lambda - \lambda_0)}{\sin (\lambda - \lambda_0)}\right] \tag{6.3.7}$$

Equations (6.3.3), (6.3.6), and (6.3.7) yield a grid such as the one in Figure 6.3.2. The meridians are straight lines, and the parallels are concentric circles. As the equator is approached the parallel circles are compressed together, and distortion becomes extreme. The plotting coordinates are in Table 6.3.1.

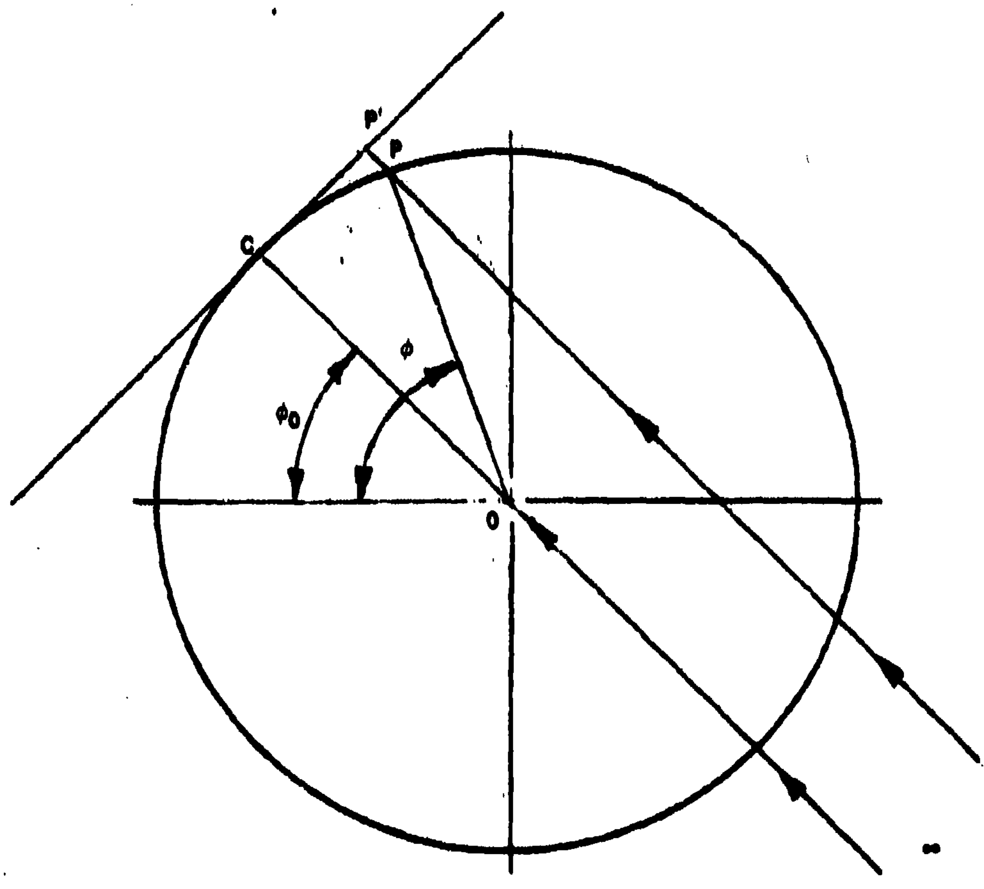


Figure 6.3.1. Geometry of the oblique orthographic projection

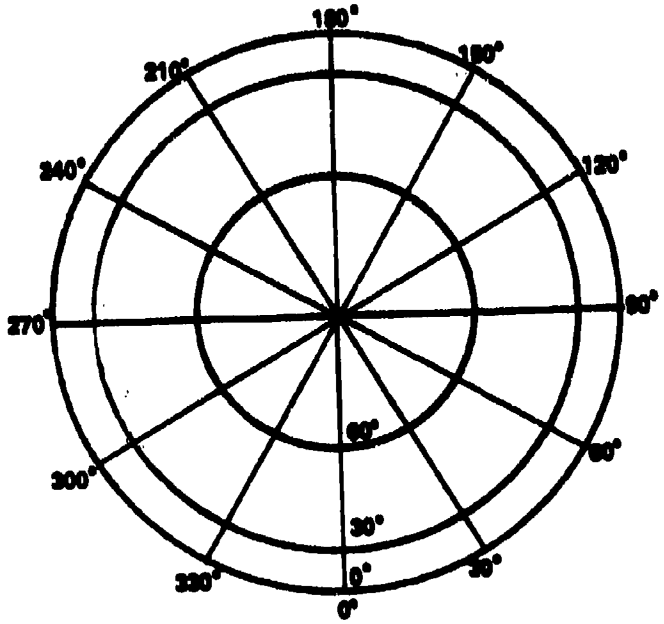


Figure 6.3.2. Orthographic polar projection

Table 8.3.1. Orthographic Projection, Polar Case.

Orthographic Polar Projection

Latitude*	Longitude*	X**	Y**
0.0000	0.0000	-.000	-6.378
0.0000	30.0000	3.189	-5.524
0.0000	60.0000	5.524	-3.189
0.0000	90.0000	6.378	.000
30.0000	0.0000	-.000	-5.524
30.0000	30.0000	2.762	-4.784
30.0000	60.0000	4.784	-2.762
30.0000	90.0000	5.524	.000
60.0000	0.0000	-.000	-3.189
60.0000	30.0000	1.595	-2.762
60.0000	60.0000	2.762	-1.595
60.0000	90.0000	3.189	-.000
90.0000	0.0000	-.000	-.000
90.0000	30.0000	-.000	-.000
90.0000	60.0000	-.000	-.000
90.0000	90.0000	-.000	-.000

$\phi_0 = 90°$ *Degrees
$\lambda_0 = 0°$ **Meters

An equatorial orthographic projection is given by Figure 6.3.3. This type of projection was used to produce the first of the Lunar maps. The grid is obtained by letting $\phi_0 = 0°$ in (6.3.4) and (6.3.5). This is another limiting case of the oblique projection.

$$\psi = \cos^{-1} [\cos\phi \cos(\lambda - \lambda_0)] \tag{6.3.8}$$

$$\theta = \tan^{-1} \left[\frac{\tan\phi}{\sin(\lambda - \lambda_0)}\right] \tag{6.3.9}$$

Equations (6.3.8) and (6.3.9) are then used with (6.3.3) to produce the required grid. In the figure, the central meridian and the equator are the only straight lines. Notice, again, from the figure that distortion becomes extreme at the margins of the map. Plotting coordinates are in Table 6.3.2.

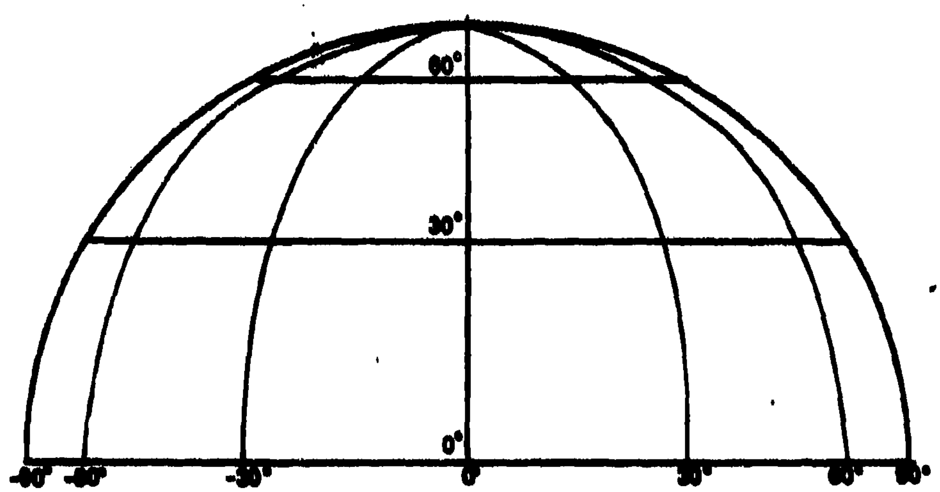


Figure 6.3.3. Orthographic equatorial projection

Table 6.3.2. Orthographic Projection, Equatorial Case.

Orthographic Equatorial Projection

Latitude*	Longitude*	X**	Y**
0.0000	0.0000	.000	.000
0.0000	30.0000	3.189	0.000
0.0000	60.0000	5.524	0.000
0.0000	90.0000	6.378	0.000
30.0000	0.0000	.000	3.189
30.0000	30.0000	2.762	3.189
30.0000	60.0000	4.784	3.189
30.0000	90.0000	5.524	3.189
60.0000	0.0000	.000	5.524
60.0000	30.0000	1.595	5.524
60.0000	60.0000	2.762	5.524
60.0000	90.0000	3.189	5.524
90.0000	0.0000	.000	6.378
90.0000	30.0000	.000	6.378
90.0000	60.0000	.000	6.378
90.0000	90.0000	.000	6.378

$\phi_0 = 0°$ *Degrees

$\lambda_0 = 0°$ **Meters

6.4 Simple Conical Projections [8], [22]

The simple conical projections to be considered in this section are the one and two standard parallel, and the perspective cases. All of these are basically graphical projections.

The geometry for the simple conical projection, with one standard parallel, is displayed in Figure 6.4.1. The cone is tangent to the sphere at the latitude ϕ_0, with central meridian at longitude λ_0. This is displayed in section, with the cone tangent at point O. The constant of the cone, from Section 1.6, is

$$c = \sin \phi_0 \tag{6.4.1}$$

From the figure

$$\rho_0 = a \cot \phi_0 \tag{6.4.2}$$

To obtain the spacing of the parallels, let the central meridian be divided truly. Thus, with the help of (6.4.2)

$$\rho = \rho_0 - a(\phi - \phi_0) \tag{6.4.3}$$

From (6.4.1)

$$\theta = c(\lambda - \lambda_0) \tag{6.4.4}$$

Substitute (6.4.1) into (6.4.4).

$$\theta = (\lambda - \lambda_0) \sin \phi_0 \tag{6.4.5}$$

The abscissa is, from (6.4.2) and (6.4.3)

$$x = aS[\cot \phi_0 - (\phi - \phi_0)] \sin [(\lambda - \lambda_0) \sin \phi_0] \tag{6.4.6}$$

The ordinate is

$$y = aS\{\cot \phi_0 - [\cot \phi_0 - (\phi - \phi_0)] \cos [(\lambda - \lambda_0) \sin \phi_0]\} \tag{6.4.7}$$

where S is the scale factor.

The grid for this projection is given in Figure 6.4.2 for $\phi_0 = 45°$ and $\lambda_0 = 0°$. All of the meridians are straight lines, and the parallels are concentric circles, equally spaced. This grid has frequently been used in atlases. Table 6.4.1 gives the plotting coordinates.

The simple conical construction for the two standard parallels case follows from Figure 6.4.3. The cone is defined to have true length standard parallels at ϕ_1 and ϕ_2, with $\phi_2 > \phi_1$. From the equal spacing criterion along the central meridian

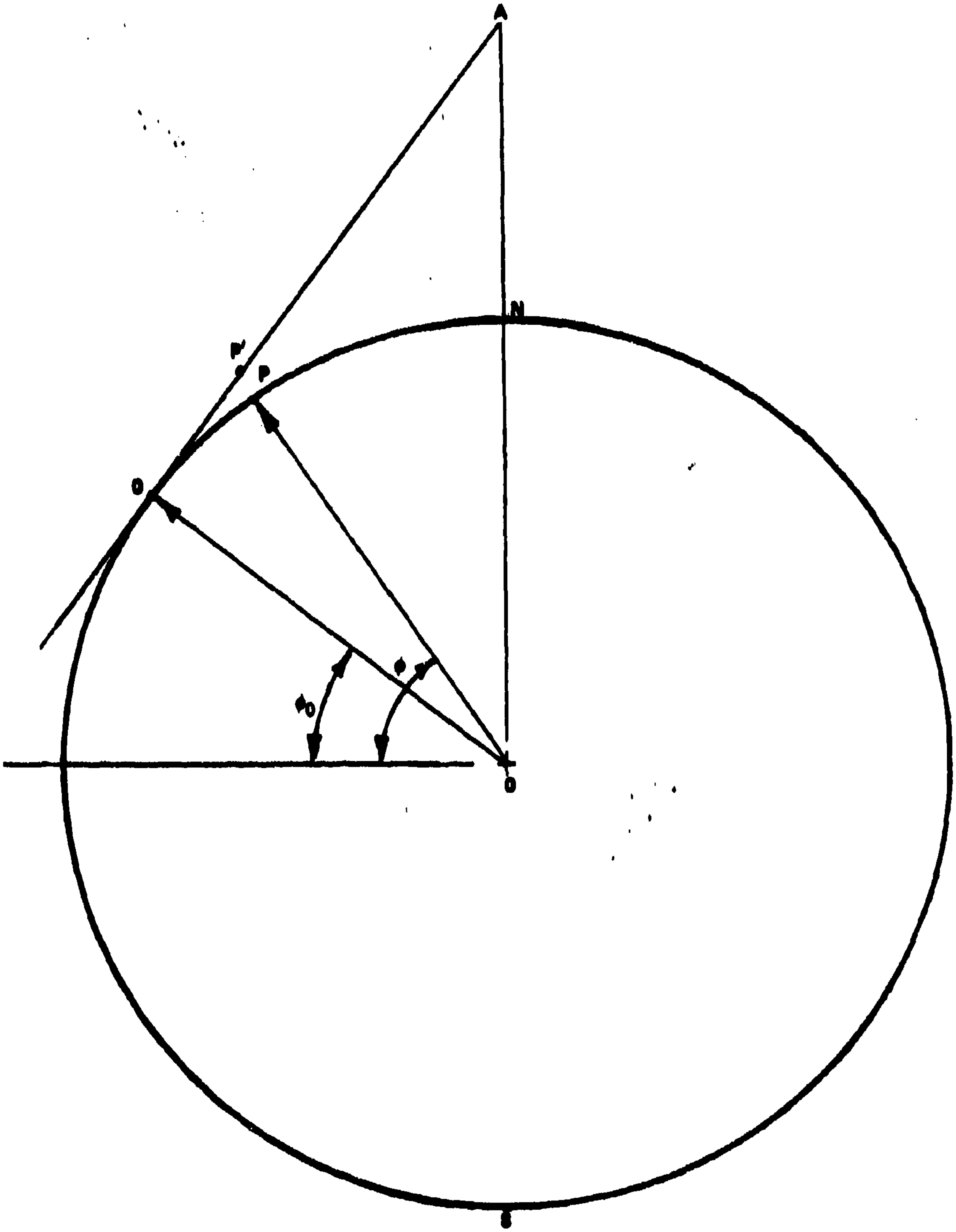


Figure 6.4.1. Geometry of the simple conical projection with one standard parallel

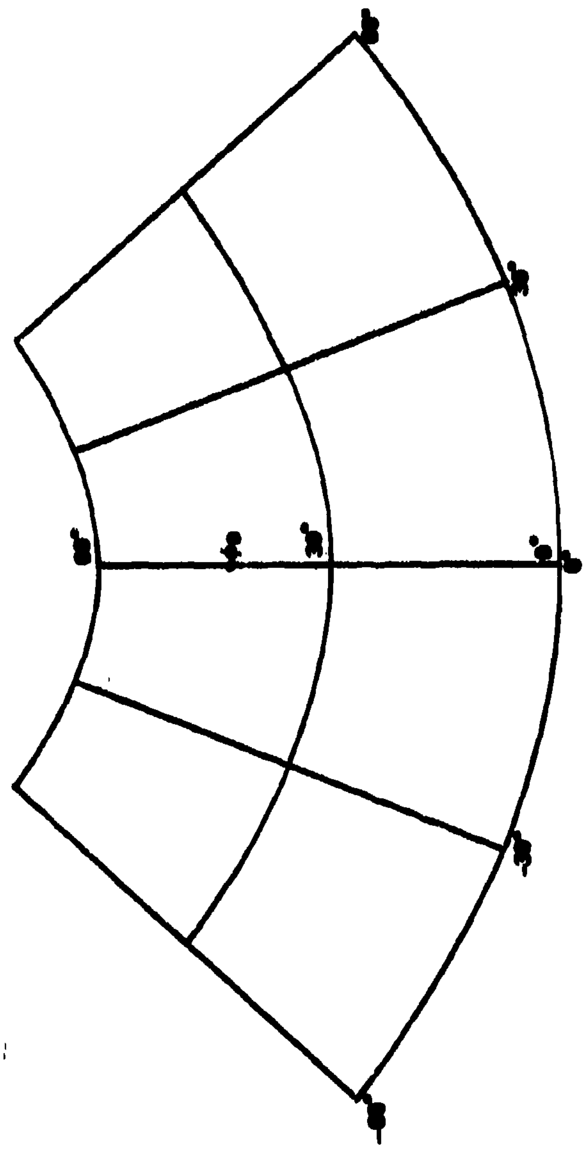


Figure 6.4.2. Simple conical projection, with one standard parallel

Table 6.4.1. Simple Conical Projection, One Standard Parallel.

Simple Conical One Standard Parallel

Latitude*	Longitude*	X**	Y**
0.0000	0.0000	0.000	-5.009
0.0000	15.0000	2.096	-4.815
0.0000	30.0000	4.121	-4.238
0.0000	45.0000	6.004	-3.298
0.0000	60.0000	7.683	-2.027
0.0000	75.0000	9.099	-.470
0.0000	90.0000	10.204	1.322
15.0000	0.0000	0.000	-3.340
15.0000	15.0000	1.789	-3.174
15.0000	30.0000	3.516	-2.681
15.0000	45.0000	5.124	-1.879
15.0000	60.0000	6.556	-.795
15.0000	75.0000	7.764	.534
15.0000	90.0000	8.707	2.063
30.0000	0.0000	0.000	-1.670
30.0000	15.0000	1.481	-1.532
30.0000	30.0000	2.912	-1.124
30.0000	45.0000	4.243	-.460
30.0000	60.0000	5.430	.438
30.0000	75.0000	6.430	1.539
30.0000	90.0000	7.211	2.805
45.0000	0.0000	0.000	.000
45.0000	15.0000	1.174	.109
45.0000	30.0000	2.308	.432
45.0000	45.0000	3.363	.959
45.0000	60.0000	4.303	1.670
45.0000	75.0000	5.096	2.543
45.0000	90.0000	5.715	3.546
60.0000	0.0000	0.000	1.670
60.0000	15.0000	.867	1.750
60.0000	30.0000	1.704	1.989
60.0000	45.0000	2.482	2.377
60.0000	60.0000	3.176	2.903
60.0000	75.0000	3.762	3.547
60.0000	90.0000	4.219	4.288

$\phi_0 = 45°$
$\lambda_0 = 0°$

*Degrees
**Meters

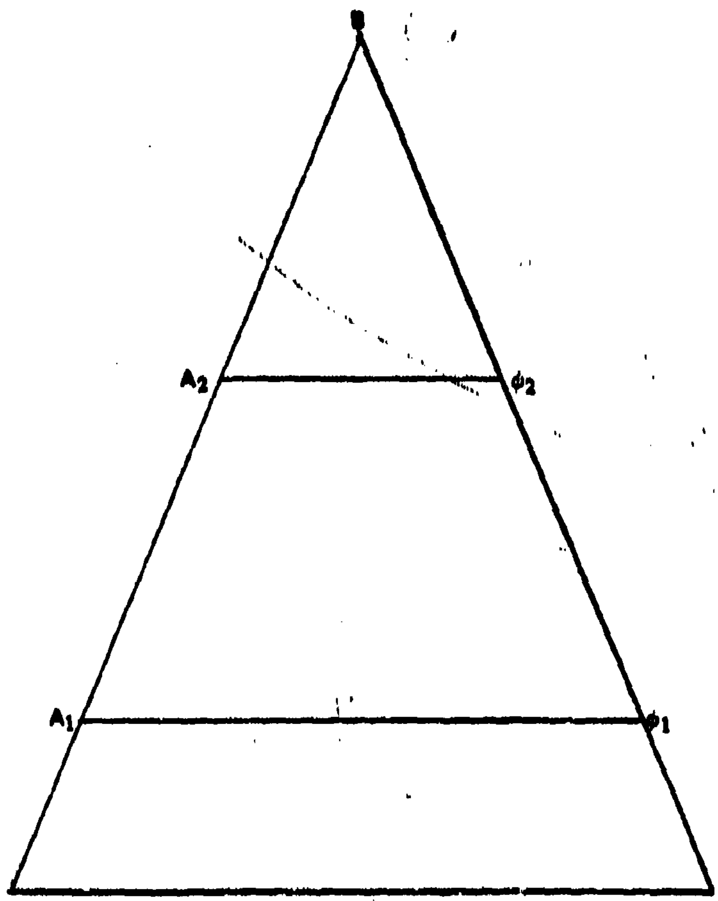


Figure 6.4.3. Geometry for the simple conical projection with two standard parallels

$$\rho_1 - \rho_2 = a(\phi_2 - \phi_1) \tag{6.4.8}$$

From the similar triangles in Figure 6.4.3

$$\frac{\rho_1}{\rho_2} = \frac{a \cos \phi_1}{a \cos \phi_2}$$

$$= \frac{\cos \phi_1}{\cos \phi_2}$$

$$\rho_2 = \rho_1 \frac{\cos \phi_2}{\cos \phi_1} \tag{6.4.9}$$

Substitute (6.4.9) into (6.4.8).

$$\rho_1 \left(1 - \frac{\cos \phi_2}{\cos \phi_1}\right) = a(\phi_2 - \phi_1)$$

$$\rho_1 = \frac{a(\phi_2 - \phi_1)}{1 - \dfrac{\cos \phi_2}{\cos \phi_1}} \tag{6.4.10}$$

A radius vector to an arbitrary point P′ of latitude ϕ on the central meridian is, with the requirement of equal spacing applied,

$$\rho = \rho_1 - a(\phi - \phi_1) \tag{6.4.11}$$

Substitute (6.4.10) into (6.4.11).

$$\rho = a\left[\frac{\phi_2 - \phi_1}{1 - \dfrac{\cos \phi_2}{\cos \phi_1}} - (\phi - \phi_1)\right] \tag{6.4.12}$$

The next step is to find a constant of the cone for this configuration. From the requirement of the circle of parallel to be true length at ϕ_1.

$$2\pi a \cos \phi_1 = 2\pi c_1 \rho_1$$

$$c_1 = \frac{a \cos \phi_1}{\rho_1} \tag{6.4.13}$$

Substitute (6.4.10) into (6.4.13)

$$c_1 = \frac{\dfrac{a \cos \phi_1}{a(\phi_2 - \phi_1)}}{1 - \dfrac{\cos \phi_2}{\cos \phi_1}}$$

$$= \cos \phi_1 \left(\frac{1 - \dfrac{\cos \phi_2}{\cos \phi_1}}{\phi_2 - \phi_1} \right)$$

$$= \frac{\cos \phi_1 - \cos \phi_2}{\phi_2 - \phi_1} \tag{6.4.14}$$

We now have the equations (6.4.12) and (6.4.14) for a polar representation of the map point. The next step is to obtain the Cartesian plotting equations. These are

$$x = aS\left\{ \frac{\phi_2 - \phi_1}{1 - \dfrac{\cos \phi_2}{\cos \phi_1}} - (\phi - \phi_1) \right\}$$

$$\times \sin [(\lambda - \lambda_0)c_1] \tag{6.4.15}$$

$$y = aS\left\{ \frac{\phi_2 - \phi_1}{1 - \dfrac{\cos \phi_2}{\cos \phi_1}} - (\phi - \phi_1) \right\}$$

$$\times \{1 - \cos [(\lambda - \lambda_0)c_1]\} \tag{6.4.16}$$

Equations (6.4.14), (6.4.15), and (6.4.15) yield the grid of Figure 6.4.4. Again, the meridians are straight lines, and the parallels are equally spaced concentric circles. This projection has been used quite often for atlas maps where it is not necessary to have either conformality or equal area. The plotting coordinates are in Table 6.4.2.

The geometry for the conical perspective projection is shown in Figure 6.4.5. The cone is tangent at latitude ϕ_0, and the central meridian has longitude λ_0.

The constant of the cone, and the radius of the parallel circle of tangency are given by (6.4.1) and (6.4.2), respectively. From the figure, the distance to an arbitrary latitude is

$$\rho = \rho_0 - a \tan (\phi - \phi_0) \tag{6.4.17}$$

We now have the polar coordinates for this projection. The Cartesian coordinates, using (6.4.1), (6.4.2), and (6.4.17) are

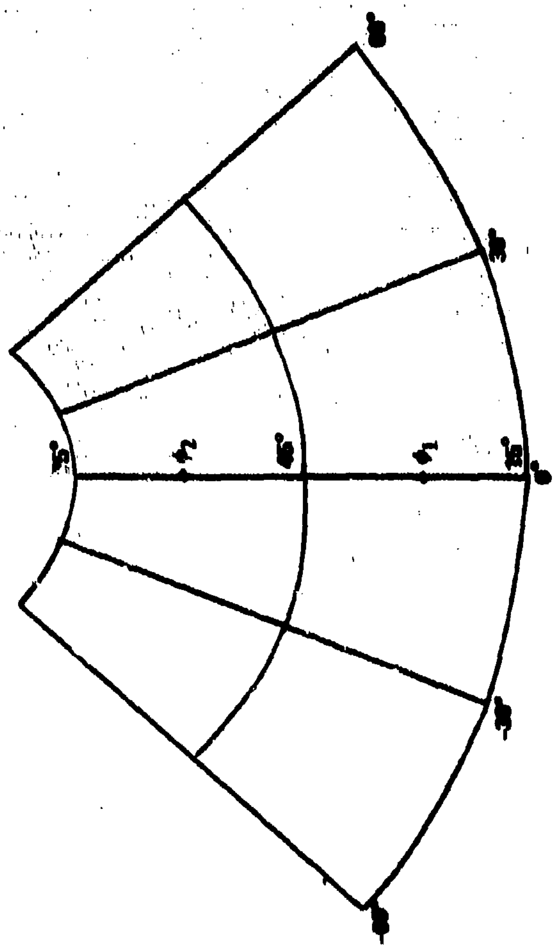


Figure 6.4.4. Simple conical projection with two standard parallels

Table 6.4.2. Simple Conical Projection, Two Standard Parallels.

Simple Conical Two Standard Parallels			
Latitude*	Longitude*	X**	Y**
15.0000	0.0000	0.000	-1.670
15.0000	15.0000	1.742	-1.510
15.0000	30.0000	3.426	-1.036
15.0000	45.0000	4.995	-.263
15.0000	60.0000	6.398	.782
30.0000	0.0000	0.000	.000
30.0000	15.0000	1.438	.132
30.0000	30.0000	2.828	.523
30.0000	45.0000	4.124	1.161
30.0000	60.0000	5.281	2.024
45.0000	0.0000	0.000	1.670
45.0000	15.0000	1.134	1.774
45.0000	30.0000	2.230	2.083
45.0000	45.0000	3.252	2.586
45.0000	60.0000	4.165	3.266
60.0000	0.0000	0.000	3.340
60.0000	15.0000	.830	3.416
60.0000	30.0000	1.633	3.642
60.0000	45.0000	2.381	4.010
60.0000	60.0000	3.049	4.508
75.0000	0.0000	0.000	5.009
75.0000	15.0000	.526	5.058
75.0000	30.0000	1.035	5.201
75.0000	45.0000	1.509	5.435
75.0000	60.0000	1.933	5.750

$\phi_1 = 30°$ *Degrees
$\phi_2 = 60°$ **Meters
$\lambda_0 = 0°$

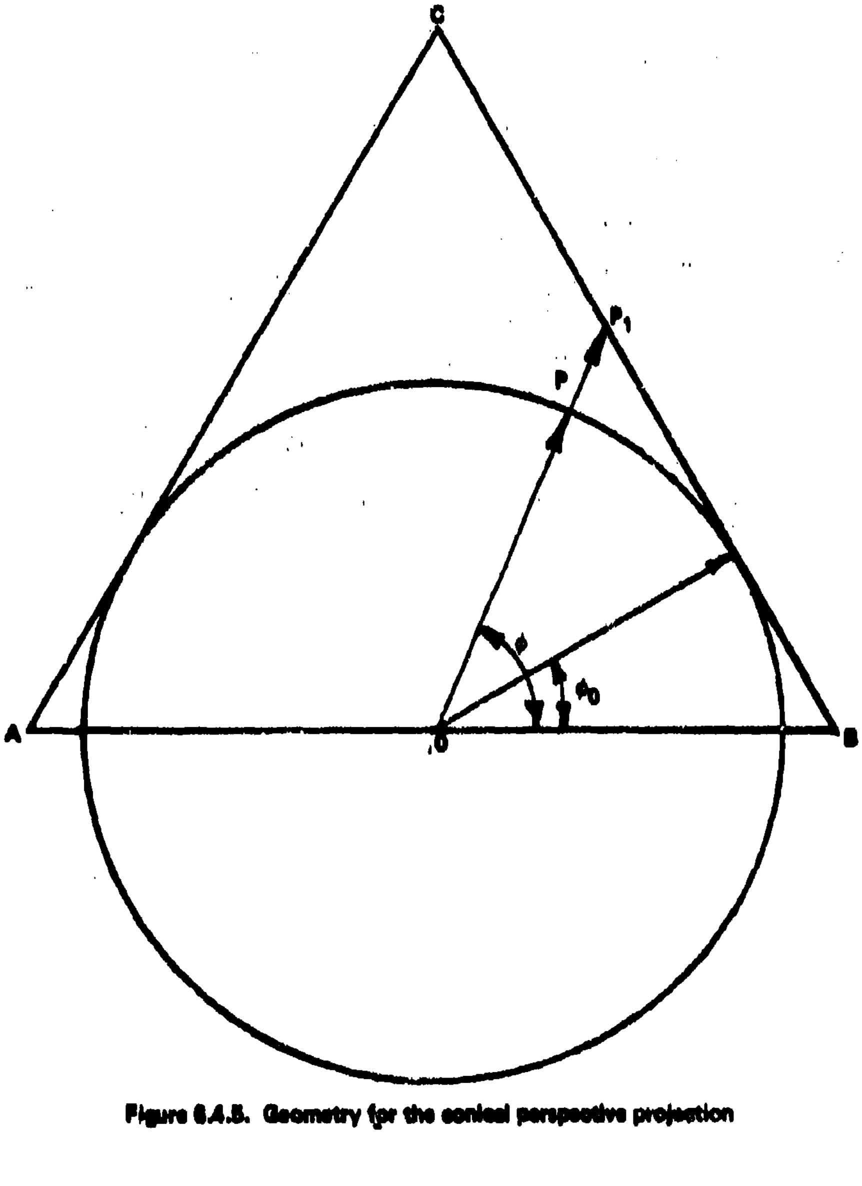


Figure 6.4.5. Geometry for the conical perspective projection

$$x = aS\,[\cot\phi_0 - \tan(\phi - \phi_0)] \times \sin[(\lambda - \lambda_0)\sin\phi_0] \tag{6.4.18}$$

$$y = aS\,[\cot\phi_0 - \tan(\phi - \phi_0)] \times \{1 - \cos[(\lambda - \lambda_0)\sin\phi_0]\} \tag{6.4.19}$$

where S is the scale factor.

Equations (6.4.18) and (6.4.19) give the grid of Figure 6.4.6. The parallels are concentric circles, and the meridians are straight lines. The spacing of the parallels increases in either direction from the standard parallel. Thus, distortion increases significantly as one moves north or south of the circle of tangency. This projection has been used for purely illustrative atlases.

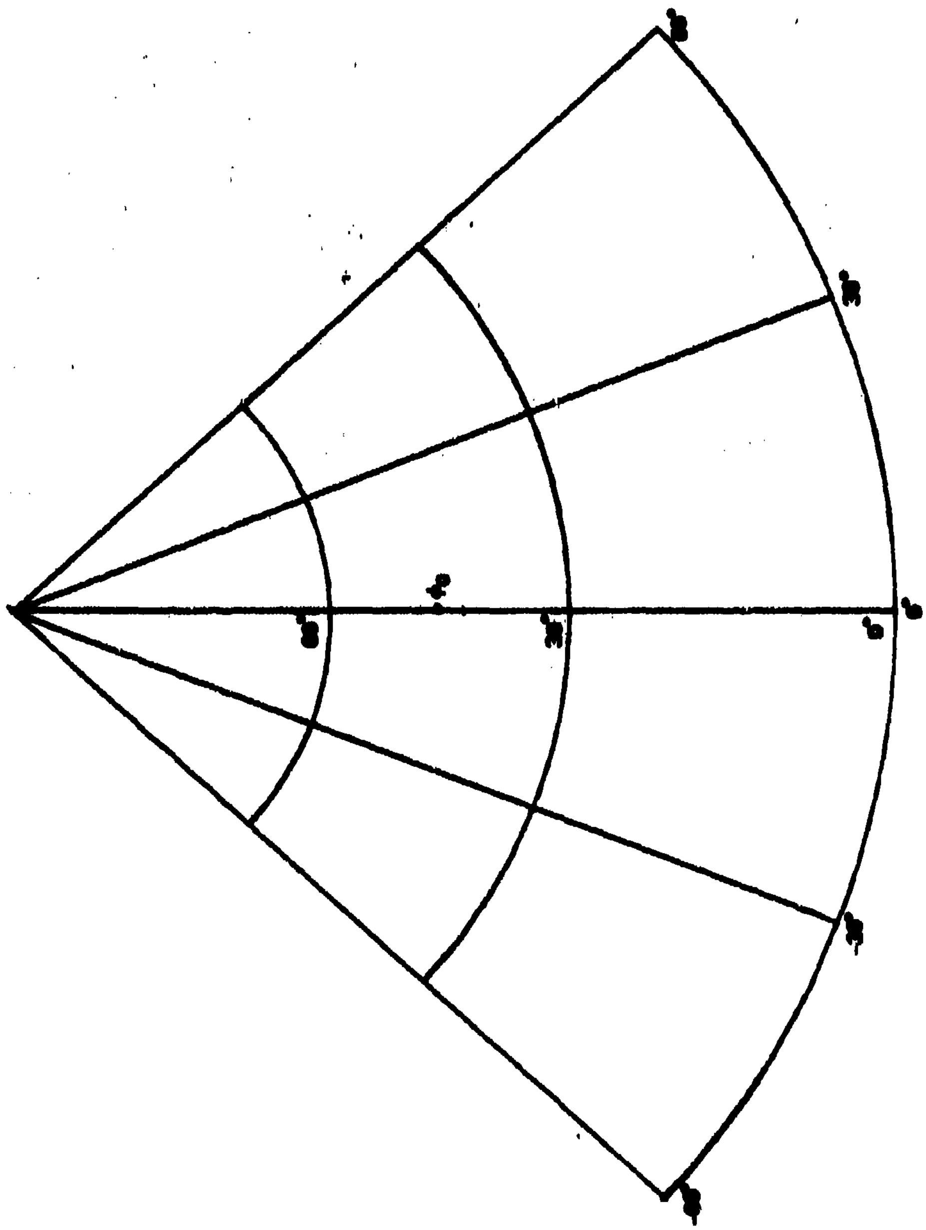


Figure 6.4.6. Perspective conical projection

6.5 Polyconic Projection [1], [25]

The polyconic projection is a modified conical projection based on a variation of the simple conical projection. The essence of the variation is that every parallel is a standard parallel, and there are an infinity of tangent cones.

First, it is necessary to derive the polar coordinates. The central meridian is true length, and has longitude λ_0. Choose some latitude ϕ_0 as the extremity of the map, and the origin of the coordinate system. Then, the distance along the meridian for a spherical earth is

$$d = a(\phi - \phi_0) \tag{6.5.1}$$

From (1.6.1), the radius to the point of tangency for an arbitrary latitude ϕ, or the first polar coordinate, is

$$\rho = a \cot \phi \tag{6.5.2}$$

and the constant of the cone is, from (1.6.4)

$$c = \sin \phi \tag{6.5.3}$$

The second polar coordinate is, from (6.5.3)

$$\theta = (\lambda - \lambda_0) \sin \phi \tag{6.5.4}$$

The Cartesian mapping equations are

$$\left.\begin{aligned} x &= \rho \sin \theta \\ y &= d + \rho(1 - \cos \theta) \end{aligned}\right\} \tag{6.5.5}$$

Substituting (6.5.1), (6.5.2), and (6.5.4) into (6.5.5).

$$\left.\begin{aligned} x &= aS \cot \phi \sin [(\lambda - \lambda_0) \sin \phi] \\ y &= aS \{\phi - \phi_0 \\ &\quad + \cot \phi [1 - \cos [(\lambda - \lambda_0) \sin \phi]]\} \end{aligned}\right\} \tag{6.5.6}$$

where S is the scale factor.

Figure 6.5.1 is the regular polyconic projection. The central meridian and the equator are the only straight lines. All other meridians are curves. The central meridian, the equator and all parallels are true length. Thus, the distortion occurs in angles, areas, and meridianal length for all meridians except for the central meridian. The polyconic projection has been used quite often in atlas and road maps, and its wide acceptance has justified its existence. A projection table for the polyconic projection is given in Table 6.5.1.

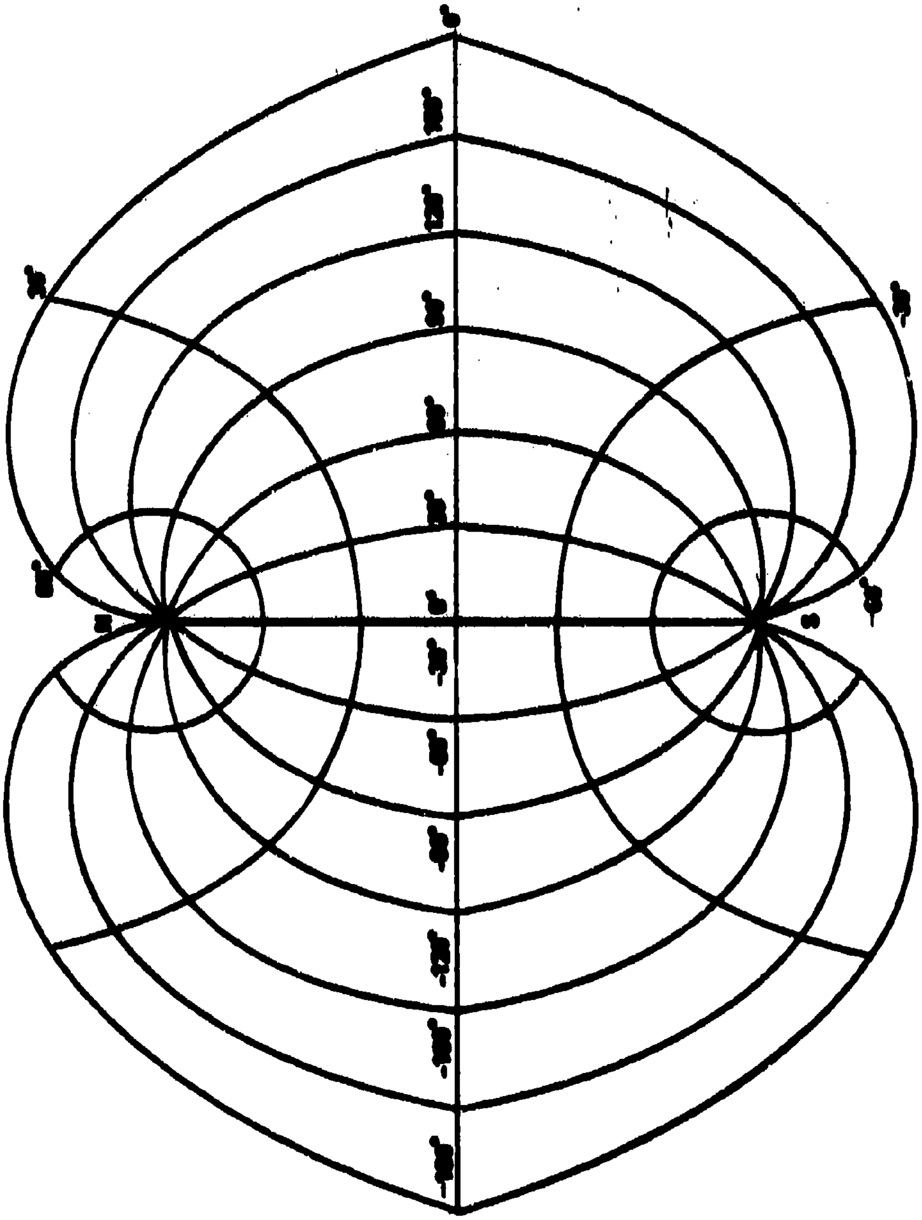


Figure 8.5.1. Regular polyconic projection

Table 6.8.1. Regular Polyconic Projection.

Regular Polyconic

Latitude*	Longitude*	X**	Y**
0.0000	0.0000	0.000	0.000
0.0000	30.0000	3.340	0.000
0.0000	60.0000	6.679	0.000
0.0000	90.0000	10.019	0.000
0.0000	120.0000	13.359	0.000
0.0000	150.0000	16.698	0.000
0.0000	180.0000	20.038	0.000
30.0000	0.0000	0.000	3.340
30.0000	30.0000	2.659	3.716
30.0000	60.0000	5.524	4.820
30.0000	90.0000	7.812	6.575
30.0000	120.0000	9.567	8.863
30.0000	150.0000	10.671	11.528
30.0000	180.0000	11.047	14.387
60.0000	0.0000	0.000	6.679
60.0000	30.0000	1.613	7.051
60.0000	60.0000	2.900	8.093
60.0000	90.0000	3.681	9.592
60.0000	120.0000	3.574	11.248
60.0000	150.0000	2.625	12.724
60.0000	180.0000	1.504	13.723
90.0000	0.0000	0.000	10.019
90.0000	30.0000	-.000	10.019
90.0000	60.0000	-.000	10.019
90.0000	90.0000	-.000	10.019
90.0000	120.0000	-.000	10.019
90.0000	150.0000	-.000	10.019
90.0000	180.0000	.000	10.019

$\lambda_0 = 0°$ *Degrees **Meters

The transverse polyconic case follows from applying the rotation formulas of Section 2.10 to the plotting equations for the regular case. Write (6.5.6) as

$$
\begin{aligned}
x &= aS \cot h \sin [(\alpha - \alpha_0) \sin h] \\
y &= aS \{h - h_0 \\
&\quad + \cot h [1 - \cos [(\alpha - \alpha_0) \sin h]]\}
\end{aligned}
\tag{6.5.7}
$$

where h, h_p, α, and α_p are measured in the auxiliary coordinate system, and S is the scale factor.

The rotation formulas, from (2.10.4), (2.10.5), and (2.10.6), for the transverse case, with 0, are

$$
\left.
\begin{aligned}
\sin h &= \cos \phi \cos (\lambda - \lambda_p) \\
\cos \alpha \cos h &= \sin \phi \\
\tan \alpha &= \sin (\lambda - \lambda_p) \cot \phi
\end{aligned}
\right\}
\tag{6.5.8}
$$

Equations (6.5.7) and (6.5.8) are used to derive grids such as the one in Figure 6.5.2. The transverse polyconic projection has also enjoyed wide acceptance in atlas and road maps.

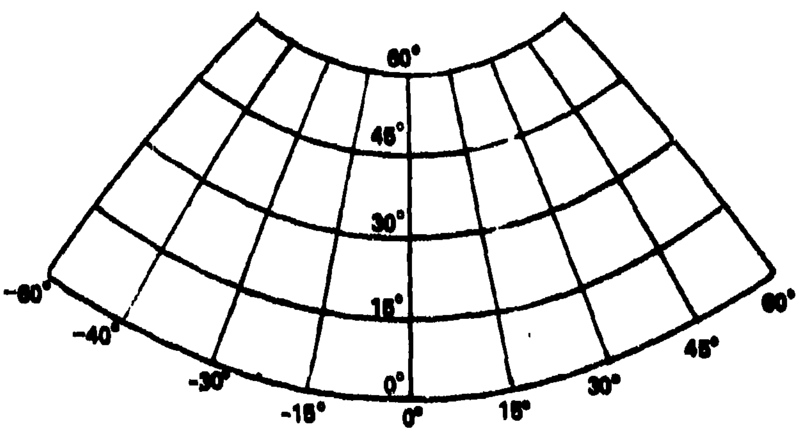


Figure 6.5.2. Transverse polyconic projection

6.6 Simple Cylindrical Projections [22]

Two simple cylindrical projections will be considered. These are the perspective and the Miller. In both cases, the sphere is transformed to the intermediate developable surface, the cylinder.

The perspective projection is a graphical representation.

Figure 6.6.1 shows the grid of the cylindrical perspective projection. The abscissa of the plotting equations is simply

$$x = aS(\lambda - \lambda_0) \tag{6.6.1}$$

where s is the scale factor, and λ_0 is the longitude of the central meridian. The ordinate follows from consideration of the figure

$$y = aS \tan \phi \tag{6.6.2}$$

Then, (6.6.1) and (6.6.2) are evaluated to obtain the grid. Distortion becomes very great as higher latitudes are reached. Thus, this projection has served in the role of an illustration.

The Miller projection calls for the equator to be 4a in length, and the meridian to be πa in length. Thus, the meridians are true, but the equator is compressed. The total area of the map is $4\pi a^2$, which is, by design, equal to the total area of the sphere. The plotting equations are simply

$$\left.\begin{aligned} x &= 2\,\frac{aS}{\pi}(\lambda - \lambda_0) \\ y &= aS\phi \end{aligned}\right\} \tag{6.6.3}$$

where λ_0 is the longitude of the central meridian, and λ, λ_0, and ϕ are in radians. Again, S is the scale factor.

The grid resulting from (6.6.3) is shown in Figure 6.6.2. In general it is better to consider this projection as conventional rather than equal area. The distortion at middle latitudes is less than in the equal area cylindrical, but it is greater at the equator. A plotting table is included as Table 6.6.1.

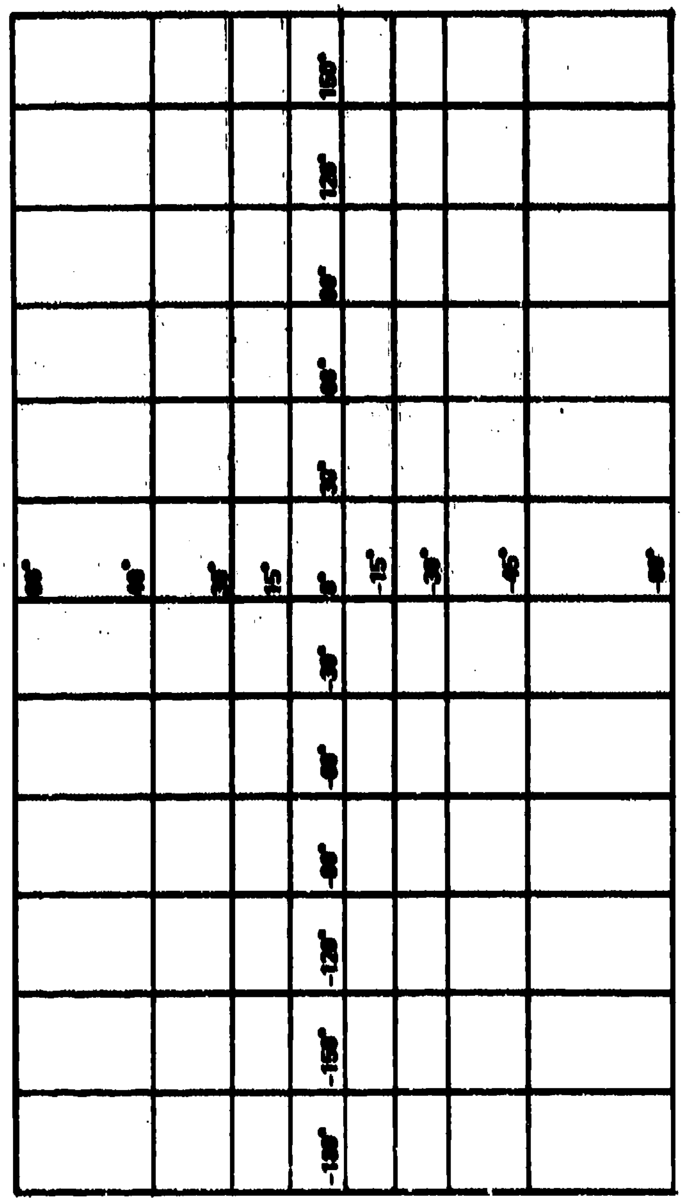


Figure 6.6.1. Perspective cylindrical projection

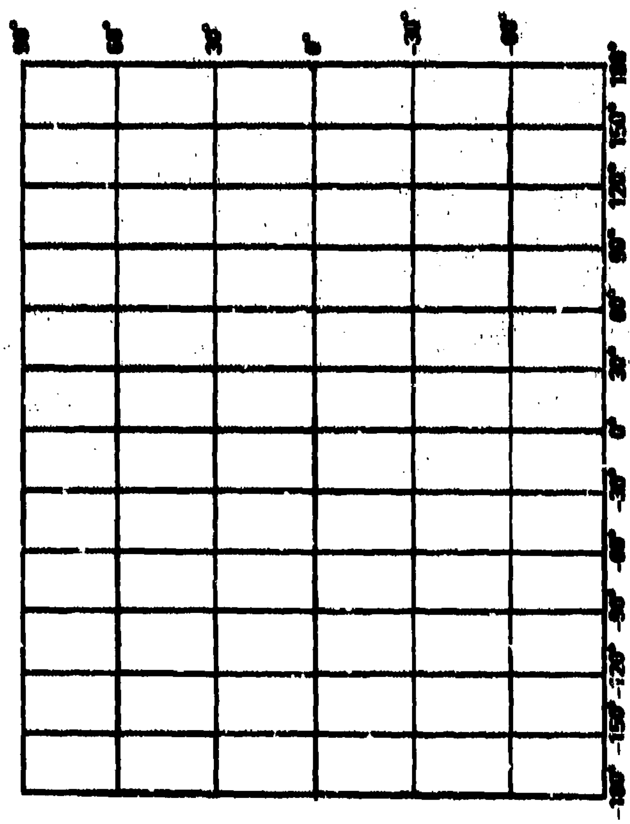


Figure 6.6.2. Miller cylindrical projection

Table 6.6.1. Simple Cylindrical Projection.

Miller Cylindrical

Latitude*	Longitude*	X**	Y**
0.0000	0.0000	0.000	0.000
0.0000	30.0000	2.126	0.000
0.0000	60.0000	4.252	0.000
0.0000	90.0000	6.378	0.000
0.0000	120.0000	8.504	0.000
0.0000	150.0000	10.630	0.000
0.0000	180.0000	12.756	0.000
30.0000	0.0000	0.000	3.340
30.0000	30.0000	2.126	3.340
30.0000	60.0000	4.252	3.340
30.0000	90.0000	6.378	3.340
30.0000	120.0000	8.504	3.340
30.0000	150.0000	10.630	3.340
30.0000	180.0000	12.756	3.340
60.0000	0.0000	0.000	6.679
60.0000	30.0000	2.126	6.679
60.0000	60.0000	4.252	6.679
60.0000	90.0000	6.378	6.679
60.0000	120.0000	8.504	6.679
60.0000	150.0000	10.630	6.679
60.0000	180.0000	12.756	6.679
90.0000	0.0000	0.000	10.019
90.0000	30.0000	2.126	10.019
90.0000	60.0000	4.252	10.019
90.0000	90.0000	6.378	10.019
90.0000	120.0000	8.504	10.019
90.0000	150.0000	10.630	10.019
90.0000	180.0000	12.756	10.019

$\lambda_0 = 0°$ *Degrees   **Meters

6.7 Plate Carrée

The Plate Carrée is a simple cylindrical projection with the equator as the standard parallel defined by a simple mathematical rule.

The meridians are true length straight lines, parallel to each other. The meridians are divided as on the sphere, so the parallels are their true distance apart. The parallels and the equator are also straight lines, perpendicular to the meridians. The equator is also divided as on the sphere.

The result of this is the square grid of Figure 6.7.1. The plotting equations which produce this grid are

$$x = aS(\lambda - \lambda_0) \qquad (6.7.1)$$

$$y = aS\phi$$

where λ_0 is the longitude of the central meridian and S is the scale factor. The angles λ, λ_0, and ϕ are in radians.

The distortion in length is extreme along the parallels. The poles, which are points in reality, are represented as straight lines. The projection pretends at neither conformality nor equivalence of area. It does serve as a reasonable diagrammatic representation of data, and is found in many technical reports where no greater cartographic sophistication is required. The Plate carrée is a map done on standard graph paper.

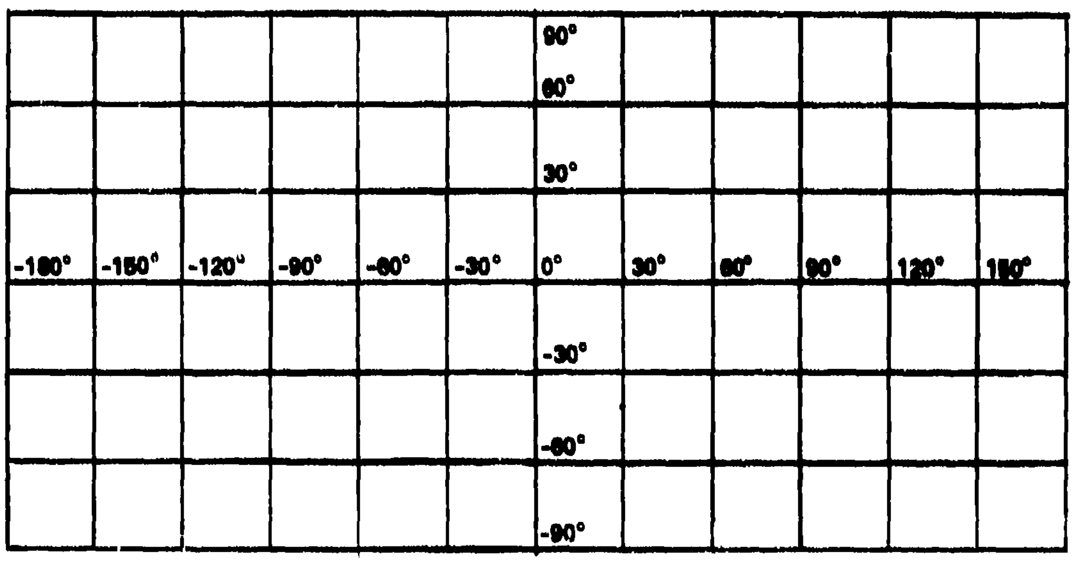


Figure 6.7.1. Plate carree projection

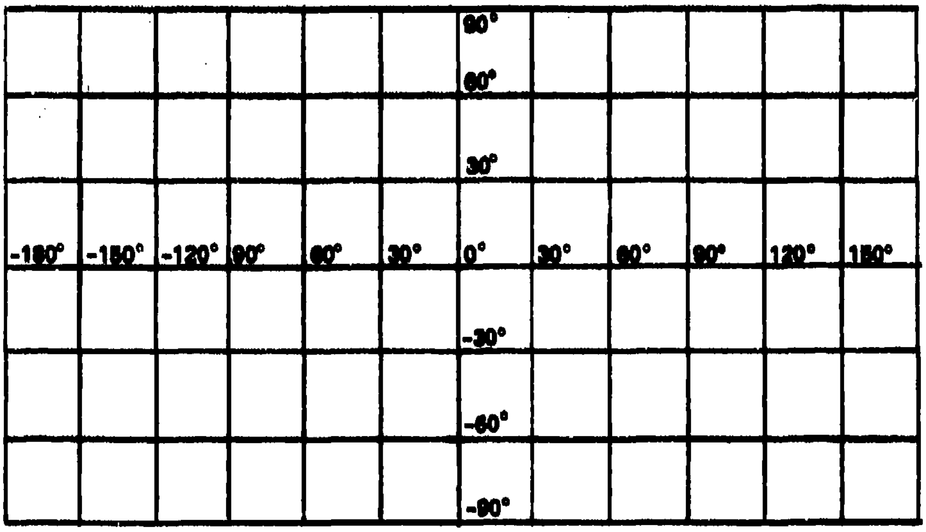
6.8 Carte Parallelogrammatique

The carte parallelogrammatique, or die reckteckige plattkarte, has a fancy name for a simple projection. It is essentially a variation for the plate carrée, in which two standard parallels, equally spaced around the equator, are taken as true length. The meridians are also true length. The plotting equations, in which ϕ_0 is the latitude of the standard parallels, and λ_0 is the longitude of the central meridian, are

$$\begin{aligned} x &= aS \cos \phi_0 \cdot (\lambda - \lambda_0) \\ y &= aS\phi \end{aligned} \qquad (6.8.1)$$

in which ϕ, λ, and λ_0 are in radians, and S is the scale factor.

A grid developed from (6.8.1) is given in Figure 6.8.1. This projection has seen some limited use in atlases. It was developed as a means of reducing some of the distortion inherent in the plate carrée. The area between the standard parallels is smaller, and that poleward from each standard parallel is larger than on the earth.



$\phi_0 = 30°$

Figure 6.8.1. Carte parallelogrammatique projection

6.9 Globular Projection [8], [22]

The globular projection is a convential means of portraying a hemisphere within a circle. In this projection, the central meridian, with longitude λ_0, and one half of the equator are diameters of the circle. The central meridian and the equator are divided truly.

Define a typical circle of parallel of latitude, ϕ. This is shown in Figure 6.9.1. Let d be the distance from the equator to the specified latitude along the central meridian. Let c be the chord length, and h be the distance from the circular arc to the chord. Let ρ be the polar radius vector.

The distance along the central meridian, between the equator, and the circle of latitude is

$$d = a\phi \tag{6.9.1}$$

From the figure

$$c = 2a\cos\phi \tag{6.9.2}$$

$$h = a\sin\phi - a\phi \tag{6.9.3}$$

From the geometry of the circular segment

$$c = \sqrt{4h(2\rho - h)} \tag{6.9.4}$$

Substitute (6.9.2), and (6.9.3) into (6.9.4), and re-arrange, to obtain the first polar coordinate, ρ.

$$c^2 = 4h(2\rho - h)$$

$$2\rho - h = \frac{c^2}{4h}$$

$$2\rho = h + \frac{c^2}{4h}$$

$$\rho = \frac{1}{2}\left(h + \frac{c^2}{4h}\right)$$

$$= \frac{1}{2}\left[a(\sin\phi - \phi) + \frac{4a^2\cos^2\phi}{4a(\sin\phi - \phi)}\right]$$

$$= \frac{a}{2}\left[\sin\phi - \phi + \frac{\cos^2\phi}{\sin\phi - \phi}\right] \tag{6.9.5}$$

The second polar coordinate, θ, follows from the geometry of the circular segment, also.

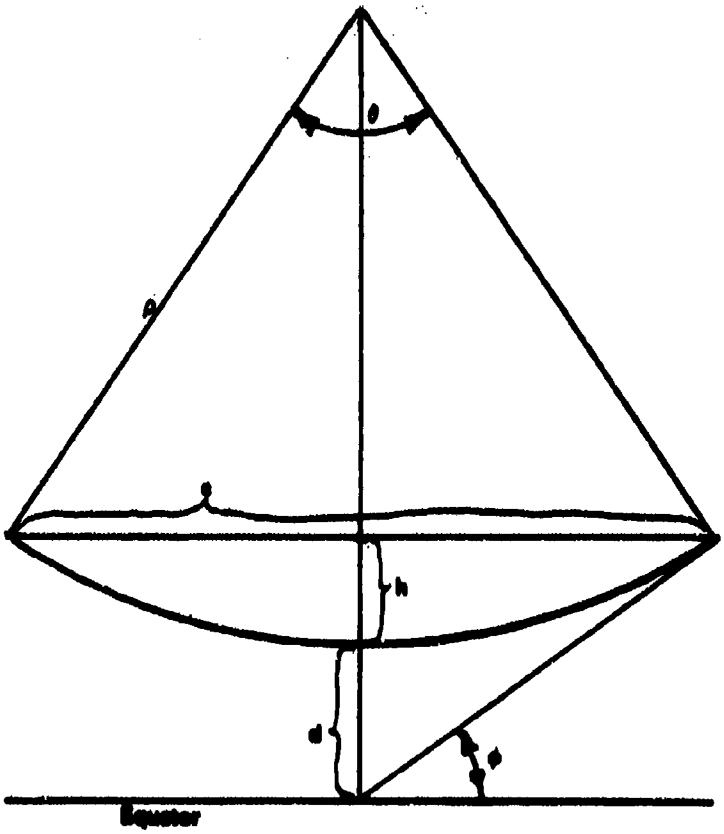


Figure 6.9.1. *Geometry for the globular projection*

$$\theta = 2 \tan^{-1} \left(\frac{c}{2h}\right) \tag{6.9.6}$$

Substitute (6.9.2) and (6.9.3) into (6.9.6)

$$\theta = 2 \tan^{-1} \left[\frac{2a \cos \phi}{a (\sin \phi - \phi)} \right]$$

$$= 2 \tan^{-1} \left[\frac{2 \cos \phi}{\sin \phi - \phi} \right] \tag{6.9.7}$$

The parallels are divided equally. Thus, a second auxiliary angle ψ can be defined by the projection

$$\frac{\psi}{\lambda - \lambda_0} = \frac{\theta}{\pi}$$

$$\psi = (\lambda - \lambda_0)\theta/\pi \tag{6.9.8}$$

Substitute (6.9.8) into (6.9.7).

$$\psi = 2(\lambda - \lambda_0) \tan^{-1} \left(\frac{2 \cos \phi}{\sin \phi - \phi}\right) \tag{6.9.9}$$

The cartesian plotting equations are

$$\left.\begin{aligned} x &= \rho \sin \psi \\ y &= d + \rho(1 - \cos \psi) \end{aligned}\right\} \tag{6.9.10}$$

Substitute (6.9.1), (6.9.5) and (6.9.9) into (6.9.10).

$$x = \frac{a}{2} \left[\sin \phi - \phi + \frac{\cos^2 \phi}{\sin \phi - \phi}\right]$$

$$\times \sin\left[2(\lambda - \lambda_0) \tan^{-1} \left(\frac{2 \cos \phi}{\sin \phi - \phi}\right)\right] \cdot S \tag{6.9.11}$$

$$y = \left\{ a\phi + \frac{a}{2}\left[\sin \phi - \phi + \frac{\cos^2 \phi}{\sin \phi - \phi}\right]\right.$$

$$\left. \times \left\{1 - \cos\left[2(\lambda - \lambda_0) \tan^{-1} \left(\frac{2 \cos \phi}{\sin \phi - \phi}\right)\right]\right\}\right\} \cdot S \tag{6.9.12}$$

where S is the scale factor.

Equations (6.9.11) and (6.9.12) produced the grid in Figure 6.9.2. The globular projection has been used for atlas maps.

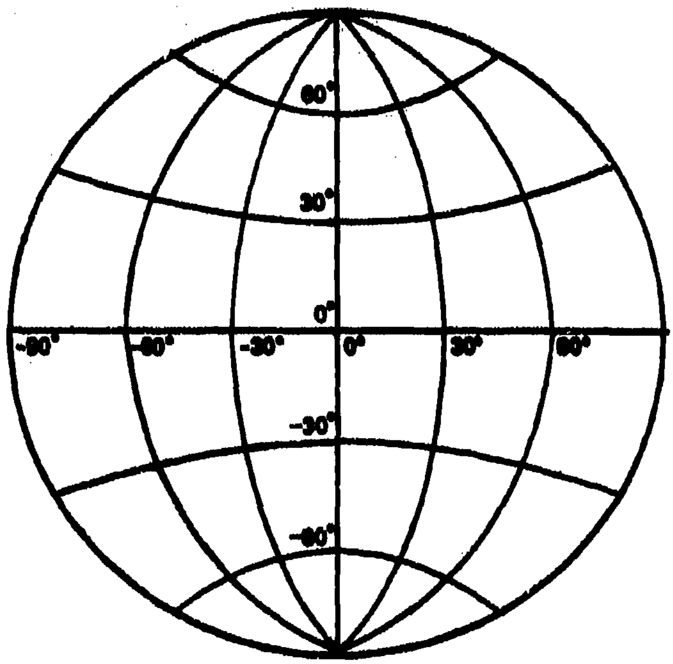


Figure 6.9.2. Globular projection

6.10 Gall's Projection [22]

Gall's projection is a stereographic cylindrical projection, with two standard parallels at 45° north and south. Figure 6.10.1 shows the geometry for the development. The meridians are spaced truly on the two standard parallels. Thus, the abscissa is

$$x = \left(aS \cos \frac{\pi}{4}\right)(\lambda - \lambda_0)$$

$$= 0.70711\ aS(\lambda - \lambda_0) \qquad (6.10.1)$$

where λ_0 is the longitude of the central meridian. The ordinates are obtained in a similar manner to the sterographic projection of Section 5.4.

$$y = 1.70711\ aS \tan \frac{\phi}{2} \qquad (6.10.2)$$

In (6.10.1) and (6.10.2), S is the scale factor, and λ and λ_0 are in radians.

This projection has been successfully used to produce world maps, since the distortion is tolerable. However, it must be kept in mind that neither conformality nor equal area is preserved.

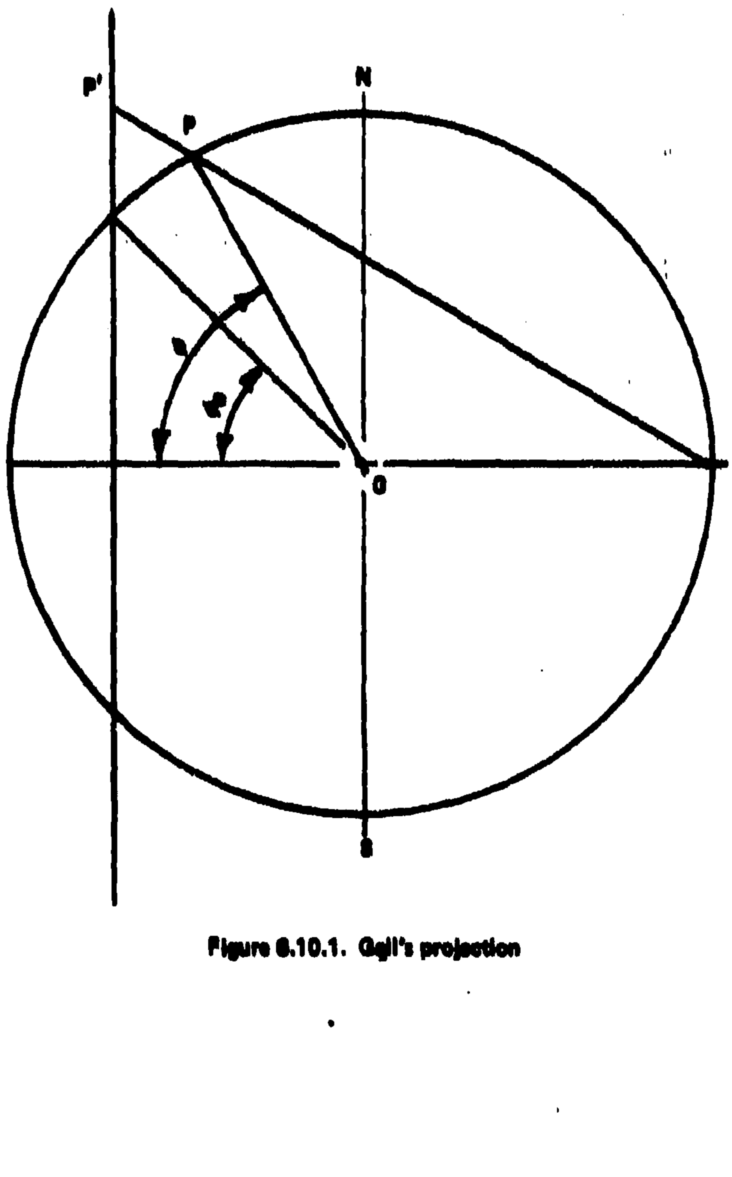


Figure 6.10.1. Gall's projection

6.11 Van der Grinten Projection [22]

The Van der Grinten projection contains the complete sphere within a circle. This projection has seen some use in atlas and National Geographic Society maps. While it does not pretend to display conformality or equal area, it does present a pleasing representation of the earth's surface. There is neither the east west extension in higher latitudes that is characteristic of the Mercator, nor the extreme compression in these areas, as shown in the sinusoidal or Mollweide.

Figure 6.11.1 gives the geometry of the projection. The equator is divided equally, and is represented by the line VQ. The line NS is the central meridian, which is also divided equally.

Consider a purely graphical construction of this projection. Join N and V. Locate an arbitrary latitude A' on NO, where

$$OA' = NO \frac{\phi}{\pi/2}$$

$$= 2NO \frac{\phi}{\pi} \qquad (6.11.1)$$

Draw AA' parallel to VQ. The intersection of AA' with NV is B. Join B and Q. The intersection of BQ and NO defines the point C'. Draw CC' parallel to VQ. Point C constitutes one of the necessary points of the projection. Its symmetric image about NO is a second such point. C" connect A and Q. AQ intersects NO at D. This is the third point necessary to completely define a circle of parallel.

A circular arc, whose radius is uniquely defined by the location of points C, D, and C", is drawn to obtain a circle of parallel.

Meridians are also circular arcs. These are fit through the poles and the equatorial point for the particular longitude. From the central meridian, of longitude λ_0, the point on the equator is

$$d = VO \frac{(\lambda - \lambda_0)}{\pi} \qquad (6.11.2)$$

The usual means of constructing a map using this projection is to inscribe the grid, and then plot points on this grid as functions of ϕ and λ.

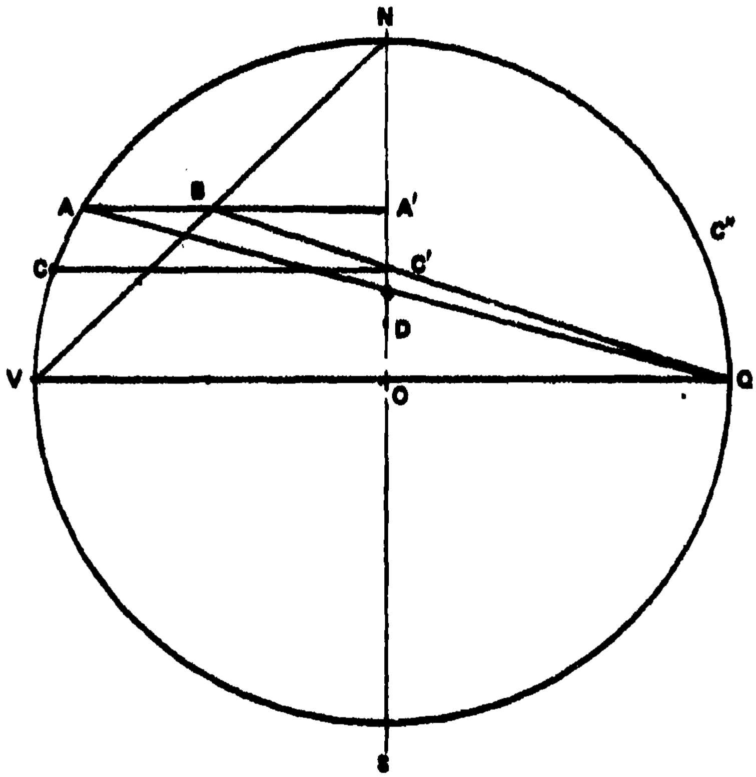


Figure 6.11.1. Van der Grinten's projection

6.12 Murdoch's Projection [22]

Murdoch's projection has been used in atlases. It is a secant projection, but differs from the simple conic with two standard parallels. There are three variations of this projection, but only one of these variations will be derived from Figure 6.12.1.

In the first variation, the parallels are spaced their true distance apart on the central meridian, with longitude λ_0. The constant of the cone is

$$c = \sin\left(\frac{\phi_1 + \phi_2}{2}\right) \tag{6.12.1}$$

where ϕ_1 is the lower standard parallel, and ϕ_2 is the upper standard parallel.

From the conditions of equal spacing

$$CB = a\left[\frac{2\sin\left(\frac{\phi_2 - \phi_1}{2}\right)}{\phi_2 - \phi_1}\right] \tag{6.12.2}$$

The middle latitude is

$$\psi = \frac{\phi_1 + \phi_2}{2} \tag{6.12.3}$$

The radius of the middle parallel is

$$TB = CB \cot \psi \tag{6.12.4}$$

The radius of the lower standard parallel is

$$\rho_1 = TB + a(\psi - \phi_1) \tag{6.12.5}$$

Thus, the first polar coordinate is

$$\rho = \rho_1 - a(\phi - \phi_1) \tag{6.12.6}$$

Substitute (6.12.2), (6.12.3), (6.12.4), and (6.12.5) into (6.12.6).

$$\rho = a\left\{\left[\frac{2\sin\left(\frac{\phi_2 - \phi_1}{2}\right)}{\phi_2 - \phi_1}\right] \cot\left(\frac{\phi_1 + \phi_2}{2}\right) - (\phi - \phi_1)\right\} \tag{6.12.7}$$

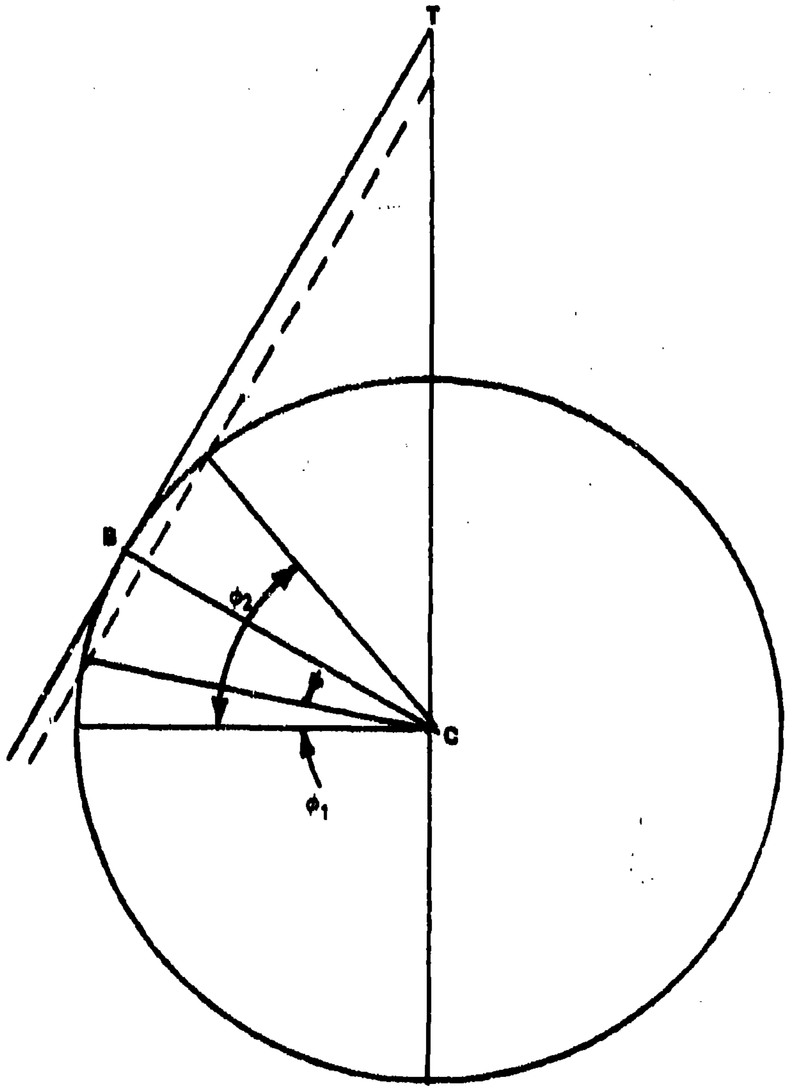


Figure 8.12.1. Geometry for the Murdoch projection

The second polar coordinate is

$$\theta = (\lambda - \lambda_0)c \tag{6.12.8}$$

Substitute (6.12.1) into (6.12.8).

$$\theta = (\lambda - \lambda_0) \sin\left(\frac{\phi_1 + \phi_2}{2}\right) \tag{6.12.9}$$

The cartesian plotting coordinates are

$$\begin{aligned} x &= [\rho \sin \theta]\,S \\ y &= [\rho_1 - \rho(1 - \cos \theta)]\,S \end{aligned} \tag{6.12.10}$$

Equations (6.12.5), (6.12.7), (6.12.9) and (6.12.10) produce the desired grid, where S is the scale factor.

6.13 Stereographic Variations [22]

Several variations of the stereographic projection have been developed to reduce distortion in regions of particular interest. These are the James, the La Hire, and the Clarke. All of these are geometric perspective projections, and can easily be obtained by an alteration of the stereographic projection of the sphere. Figure 6.13.1 shows the location of the projection points for a polar projections of these variations as compared to those of the gnomonic and the stereographic. La Hire took the projection point as 1.71 times the radius of the earth. James used 1.367 times the radius, and Clarke's value varied between 1.35 and 1.65 times the radius. We will consider a derivation of the plotting equations for a polar projection.

From Figure 6.13.2,

$$\tan \psi = \frac{\rho}{a(1 + c')}$$

$$\rho = a(1 + c') \tan \psi \tag{6.13.1}$$

It is now necessary to relate ψ and ϕ.

$$AB = a \sin (90^\circ - \phi)$$

$$= a \cos \phi \tag{6.13.2}$$

$$\tan \psi = \frac{AB}{a\,[\cos (90^\circ - \phi) + c']}$$

$$= \frac{AB}{a\,(\sin \phi + c')} \tag{6.13.3}$$

Substitute (6.13.2) into (6.13.3)

$$\tan \psi = \frac{a \cos \phi}{a\,(\sin \phi + c')}$$

$$= \frac{\cos \phi}{\sin \phi + c'} \tag{6.13.4}$$

Substitute (6.13.4) into (6.13.1).

$$\rho = \frac{a(1 + c') \cos \phi}{\sin \phi + c'} \tag{6.13.5}$$

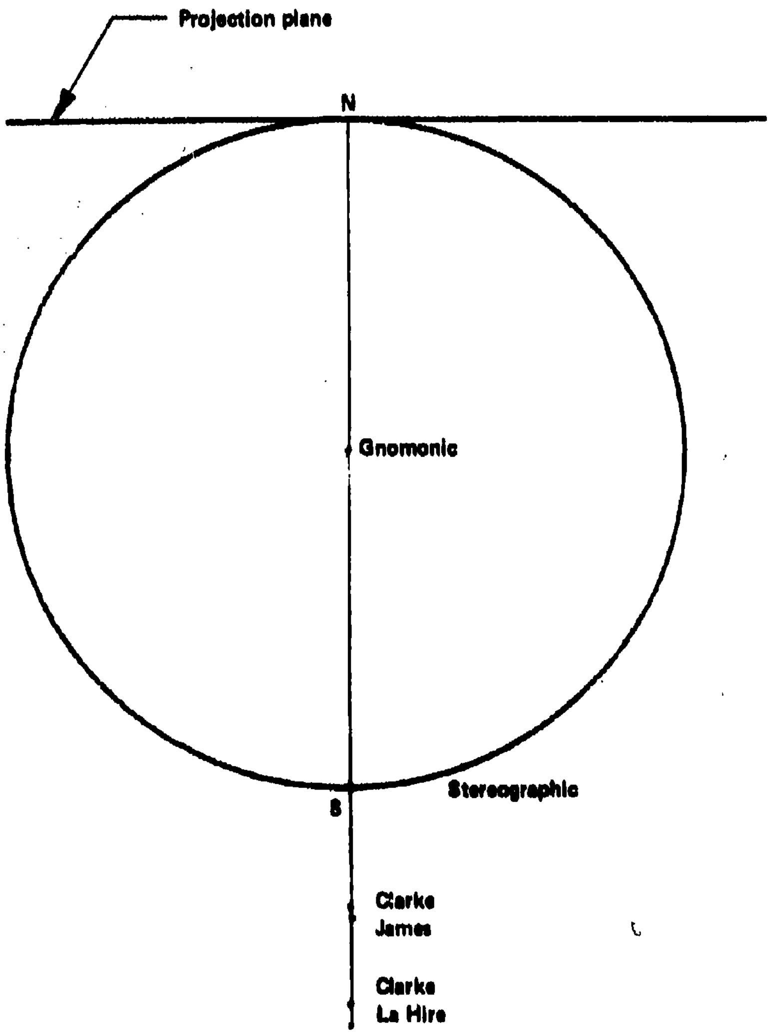


Figure 6.13.1. Projection points for the stereographic variations of the polar projection

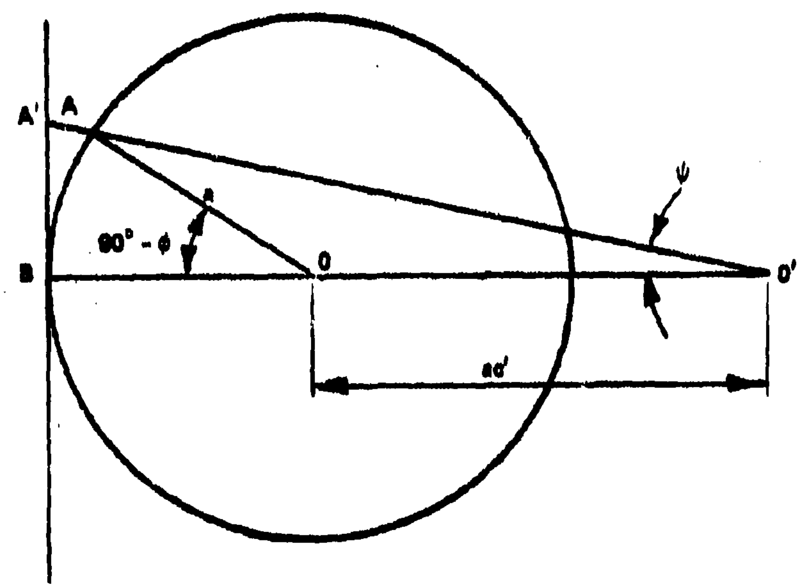


Figure 8.13.2. Geometry for the stereographic variations

The Cartesian plotting equations follow from (6.13.5)

$$x = \frac{aS(1 + c') \cos \phi}{\sin \phi + c'} \cos (\lambda - \lambda_0)$$

$$y = \frac{aS(1 + c') \cos \phi}{\sin \phi + c'} \sin (\lambda - \lambda_0) \tag{6.13.6}$$

where S is the scale factor, and c′ is given by Table 6.13.1.

This series of projections has had some utility in geodetic mapping.

Table 6.13.1. Values of c′ for the variations of the Stereographic Projection.

Name	c′
Clarke	1.35
James	1.367
Clarke	1.65
La Hire	1.71

6.14 Cassini's Projection [22]

Cassini projection is, in effect, a transverse cylindrical equidistant projection. For this projection, choose a central meridian, and let the origin be at the intersection of this central meridian and the equator. The mapping coordinates follow simply from a consideration of distance on a sphere.

In Figure 6.14.1, P is the arbitrary point as the sphere, with latitude ϕ and longitude λ. PQ is a great circle through P and perpendicular to the central meridian.

Let ψ and θ be auxiliary central angles, as shown in the figure. Apply Napier's rules to the spherical triangle to obtain these angles.

$$\begin{aligned}\sin\psi &= \cos(90^\circ - \lambda)\cos\phi \\ &= \sin\lambda\cos\phi \\ \psi &= \sin^{-1}(\sin\lambda\cos\phi)\end{aligned} \tag{6.14.1}$$

$$\sin(90^\circ - \lambda) = \tan\phi\tan\theta$$

$$\tan\theta = \frac{\cos\lambda}{\tan\phi}$$

$$\theta = \tan^{-1}\left(\frac{\cos\lambda}{\tan\phi}\right) \tag{6.14.2}$$

The mapping equations follow from (6.14.1) and (6.14.2) plus the radius of the sphere.

$$\left.\begin{aligned}x &= a\cdot S\cdot\sin^{-1}(\sin\lambda\cos\phi) \\ y &= a\cdot S\cdot\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{\cos\lambda}{\tan\phi}\right)\right]\end{aligned}\right\} \tag{6.14.3}$$

where S is the scale factor.

From (6.14.3), the true distance on the sphere is maintained, as is also the perpendicularity.

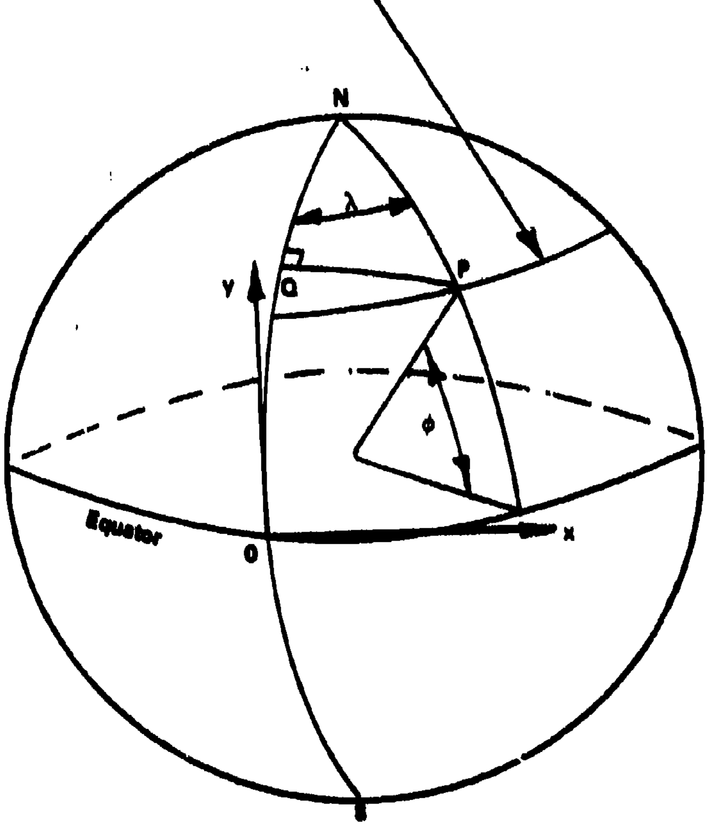


Figure 6.14.1. Geometry for the Cassini projection

Chapter 7

THEORY OF DISTORTIONS

Distortion is the cartographic bane. This was mentioned in Chapter 1. No matter what technique or algorithm is used, distortion will occur in length, angle or area, or in a combination of these. Throughout Chapters 4, 5, and 6, it was pointed out where distortions do occur for particular projection, but this was treated only qualitatively. This brief chapter will deal with the general theory of distortion in maps in a quantitative way. Formulas will be developed to quantify the distortions in length, angle, and area. Then, these generalities will be applied to the most frequently used equal area, conformal, and conventional projections. Thus, we will have suggested a numerical means of assessing the acceptability of a map for a particular application. Finally, some of the different classes of maps will be compared in a qualitative manner.

7.1 Distortion in Length [10], [20]

In order to describe distortion in length, we will consider a two dimensional plotting surface, and derive terms for distortion along the parallels and meridians, as compared to true distance along a sphere or spheroid. The derivation begins with the first fundamental forms of the earth and the plotting surface. The ratio of these fundamental forms is defined to be m. From (2.3.3)

$$m^2 = \frac{E(d\phi)^2 + 2\,F\,d\phi\,d\lambda + G(d\lambda)^2}{e(d\phi)^2 + 2\,f\,d\phi\,d\lambda + g(d\lambda)^2} \tag{7.1.1}$$

where the capital letters refer to the mapping surface, and the lower case, to the sphere or spheroid. Since we are dealing exclusively in orthogonal systems for the mapping surface and the earth, the substitution of $F = f = 0$ into (7.1.1) yields

$$m^2 = \frac{E(d\phi)^2 + G(d\lambda)^2}{e(d\phi)^2 + g(d\lambda)^2} \tag{7.1.2}$$

The distortion along the parametric ϕ-curve, or meridian, where $d\lambda = 0$, is, from (7.1.2).

$$m_m = \sqrt{\frac{E}{e}} \tag{7.1.3}$$

and the distortion along the parametric λ-curve, perpendicular to the meridian, where $d\phi = 0$, is

$$m_p = \sqrt{\frac{G}{g}} \qquad (7.1.4)$$

These distortions in distance will be applied to particular projections in Sections 7.4, 7.5, and 7.6. True length corresponds to $m_p = m_m = 1$.

7.2 Distortions in angles [10], [20]

Figure 7.2.1a is needed for the derivation of the angular distortion. Again, from the first fundamental for the model of the earth

$$(ds)^2 = e(d\phi)^2 + 2f\,d\phi\,d\lambda + g(d\lambda)^2. \qquad (7.2.1)$$

The angle between the parametric ϕ-, and λ-curves intersecting at point P is

$$\omega = \alpha + \beta \qquad (7.2.2)$$

Consider the differential parallelogram to be sufficiently small in area, so that it can be treated as a plane. Then, the law of cosines applies.

$$\begin{aligned}(ds)^2 &= e(d\phi)^2 + g(d\lambda)^2 \\ &\quad + 2\sqrt{eg}\,d\phi\,d\lambda\cos\omega \qquad (7.2.3)\end{aligned}$$

Equating (7.2.1) and (7.2.3)

$$\cos\omega = \frac{f}{\sqrt{eg}} \qquad (7.2.4)$$

Also,

$$\sin\omega = \sqrt{\frac{eg - f^2}{eg}} \qquad (7.2.5)$$

Since we will deal with orthogonal systems, $f = 0$, and from (7.2.4) and (7.2.5)

$$\begin{aligned}\cos\omega &= 0 \\ \sin\omega &= 1 \\ \omega &= \pi/2\end{aligned}$$

Thus, from (7.2.2)

$$\alpha + \beta = \pi/2 \qquad (7.2.6)$$

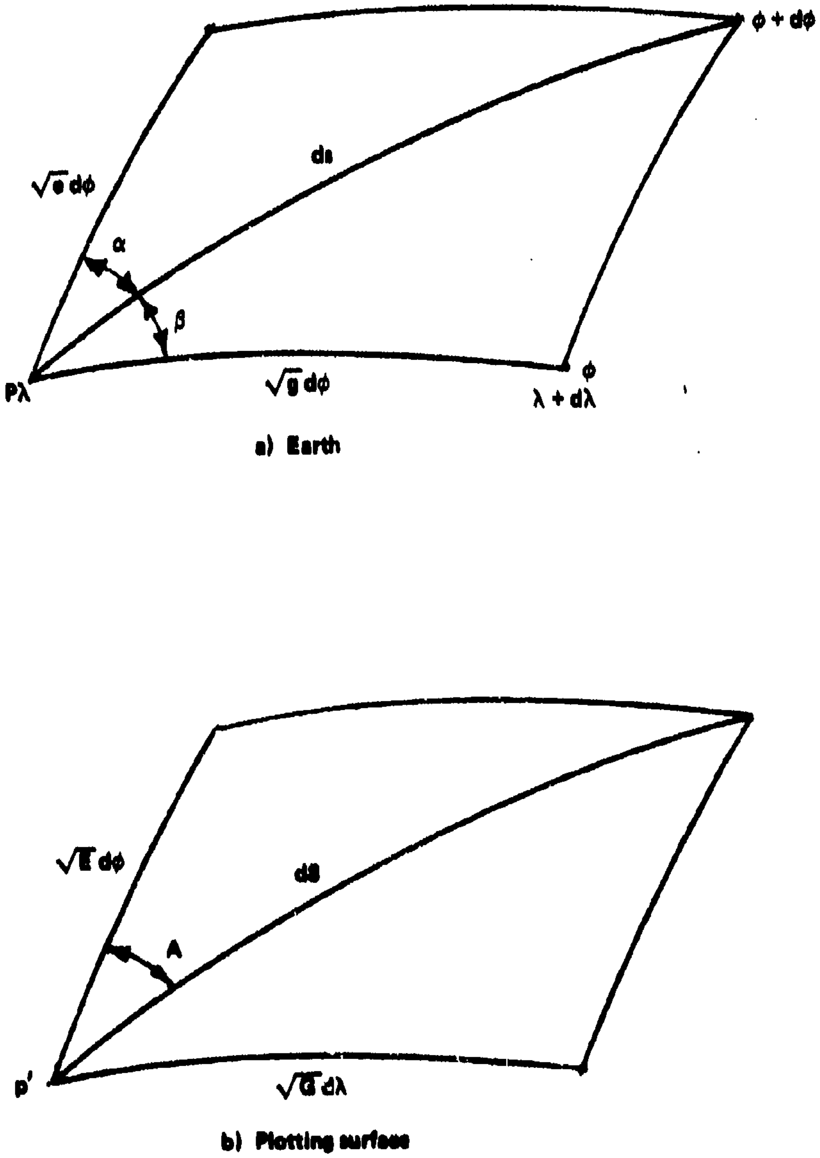


Figure 7.2.1. Differential parallelogram

From the figure

$$\left.\begin{aligned}\cos\alpha &= \sqrt{e}\,\frac{d\phi}{ds}\\ \sin\alpha &= \sqrt{g}\,\frac{d\lambda}{ds}\end{aligned}\right\} \qquad (7.2.7)$$

Consider now, Figure 7.2.1b, where an infinitesimal part of the projection surface is shown at point P′, corresponding to point P on the earth. From the figure, for the orthogonal mapping surface,

$$\left.\begin{aligned}\cos A &= \sqrt{E}\,\frac{d\phi}{dS}\\ \sin A &= \sqrt{G}\,\frac{d\lambda}{dS}\end{aligned}\right\} \qquad (7.2.8)$$

Expand

$$\begin{aligned}\sin(A-\alpha) &= \sin A\cos\alpha\\ &\quad - \cos A\sin\alpha\end{aligned} \qquad (7.2.9)$$

Substitute (7.2.7) and (7.2.8) into (7.2.9).

$$\begin{aligned}\sin(A-\alpha) &= \sqrt{Ge}\,\frac{d\lambda}{dS}\frac{d\phi}{ds} - \sqrt{Eg}\,\frac{d\lambda}{ds}\frac{d\phi}{dS}\\ &= (\sqrt{Ge} - \sqrt{Eg})\,\frac{d\phi}{ds}\frac{d\lambda}{dS}\end{aligned} \qquad (7.2.10)$$

Substitute (7.1.3) and (7.1.4) into (7.2.10).

$$\sin(A-\alpha) = (m_p - m_m)\,\frac{d\phi}{ds}\frac{d\lambda}{dS}\sqrt{eg} \qquad (7.2.11)$$

In a similar expansion,

$$\sin(A+\alpha) = (m_p + m_m)\,\frac{d\phi}{ds}\frac{d\lambda}{dS}\sqrt{eg} \qquad (7.2.12)$$

Re-arrange (7.2.11) and (7.2.12), and equate

$$\sin(A-\alpha) = \frac{m_p - m_m}{m_p + m_m}\,\sin(A+\alpha) \qquad (7.2.13)$$

Equation (7.2.13) is another transcendental beast. For a constant m_p and m_m it can be solved by the Newton-Raphson method. [14] Write (7.2.13) as

$$f(\alpha) = 0$$

$$= \sin(A-\alpha) - \frac{m_p - m_m}{m_p + m_m} \sin(A+\alpha) \tag{7.2.14}$$

The derivative of (7.2.14) is

$$\frac{df}{dA} = \cos(A-\alpha) - \frac{m_p - m_m}{m_p + m_m} \cos(A-\alpha) \tag{7.2.15}$$

The Newton-Raphson scheme for the solution of (7.2.13) is then

$$A_{n+1} = A_n - \frac{f}{\frac{df}{dA}}. \tag{7.2.16}$$

Substitute (7.2.14) and (7.2.15) into (7.2.16).

$$A_{n+1} = A_n - \frac{\sin(A_n - \alpha) - \left(\frac{m_p - m_m}{m_p + m_m}\right) \sin(A_n + \alpha)}{\cos(A_n - \alpha) - \left(\frac{m_p - m_m}{m_p + m_m}\right) \cos(A_n + \alpha)} \tag{7.2.17}$$

As an initialization, let $A_0 = \alpha$. This iteration is rapidly convergent, and easily computerized.

This technique will be applied to the equal area, and the conventional projections. As will be seen in Section 7.6, (7.2.13) has a unique solution for conformal projections.

7.3 Distortion in Area [10], [20]

The area on the map is, from (2.3.13)

$$A_m = \sqrt{EG - F^2} \tag{7.3.1}$$

and the area on the model of the earth is

$$A_e = \sqrt{eg - f^2} \tag{7.3.2}$$

The distortion of area is hereby defined to be

$$D_A = A_m / A_e$$

$$= \sqrt{\frac{EG - F^2}{eg - f^2}} \qquad (7.3.3)$$

Since the systems are orthogonal, $F = f = 0$ can be substituted into (7.3.3).

$$D_A = \sqrt{\frac{EG}{eg}} \qquad (7.3.4)$$

Substitute (7.1.3) and (7.1.4) into (7.3.4).

$$D_A = m_p \, m_m \qquad (7.3.5)$$

Now we are in a position to consider these distortions in terms of selected projections. In what follows, the distortions will be derived for polar and regular cases. They apply equally for the oblique, transverse, and equatorial cases. The only difference is that α is substituted for λ, and h is substituted for ϕ.

7.4 Distortion in Equal Area Projections [20]

Equal area projections, by their definition, have no distortion in area in the mapping transformation. Thus, from (7.3.5)

$$D_A = m_p \, m_m$$

$$= 1 \qquad (7.4.1)$$

It is seen from (7.4.1) that m_p and m_m are the reciprocals of each other. We will now consider the more important equal area projections of Chapter 4, the Albers conical, the Lambert polar azimuthal, and the cylindrical. The distortions in length and angle, at an arbitrary point, will be derived. In the case of distortions in length, only one of the distortion factors needs to be derived. Equation (7.4.1) is then used to find the seond one. Thus, in all cases, the easier of the two will be chosen for the derivation, and the second will be its reciprocal. All cases will be derived by considering the authalic sphere of Section 4.1.

Consider first the conical projection with one standard parallel, the first of the Albers projections. From (7.1.3), (4.2.1), (4.2.2), and (4.2.5)

$$m_m = \sqrt{\frac{E}{e}}$$

$$= \sqrt{\frac{c_1^2 \rho^2}{R^2 \cos^2 \phi}}$$

$$= \frac{c_1 \rho}{R \cos \phi} \tag{7.4.2}$$

Substitute (4.2.15) and (4.2.19) into (7.4.2).

$$m_m = \frac{\sin \phi_0 \dfrac{R}{\sin \phi_0} \sqrt{1 + \sin^2 \phi_0 - 2 \sin \phi \sin \phi_0}}{R \cos \phi}$$

$$= \frac{\sqrt{1 + \sin^2 \phi_0 - 2 \sin \phi \sin \phi_0}}{\cos \phi} \tag{7.4.3}$$

From (7.4.1)

$$m_p = \frac{\cos \phi}{\sqrt{1 + \sin^2 \phi_0 - 2 \sin \phi \cos \phi_0}} \tag{7.4.4}$$

It is obvious from (7.4.3) and (7.4.4) that an expansion in scale in one direction is offset by a contraction in the direction orthogonal to it.

Consider next the case of the two standard parallel Albers projection. From (4.2.6), (4.2.32) and (7.4.2)

$$m_m = \frac{(\sin \phi_1 + \sin \phi_2)}{2R \cos \phi} \sqrt{\rho_2^2 + \frac{4R^2 (\sin \phi_2 - \sin \phi)}{\sin \phi_1 + \sin \phi_2}} \tag{7.4.5}$$

From (4.2.33)

$$m_m = \frac{(\sin \phi_1 + \sin \phi_2)}{2R \cos \phi} \sqrt{\frac{4R^2 \cos^2 \phi + 4R^2 (\sin \phi_2 - \sin \phi)}{\sin \phi_1 + \sin \phi_2}}$$

$$= \sqrt{1 + \frac{(\sin \phi_1 + \sin \phi_2)(\sin \phi_2 - \sin \phi)}{\cos^2 \phi}} \tag{7.4.6}$$

Then, from (7.4.1)

$$m_p = \sqrt{\frac{\cos^2 \phi}{\cos^2 \phi + (\sin \phi_1 + \sin \phi_2)(\sin \phi_2 - \sin \phi)}} \tag{7.4.7}$$

Next, we shall consider the polar azimuthal, or Lambert, projection. In this case, $\sin \phi_0 = 1$, is substituted into (7.4.2)

$$m_m = \frac{\rho}{R \cos \phi} \tag{7.4.8}$$

Substitute (4.3.2) into (7.4.8)

$$m_m = \frac{R\sqrt{2(1-\sin\phi)}}{R\cos\phi}$$

$$= \frac{\sqrt{2(1-\sin\phi)}}{\cos\phi} \tag{7.4.9}$$

From (7.4.1)

$$m_p = \frac{\cos\phi}{\sqrt{2(1-\sin\phi)}} \tag{7.4.10}$$

The cylindrical equal area projection was treated in Section 4.5. From (7.1.3), (4.5.2), and an application of the fundamental transformation matrix

$$m_m = \sqrt{\frac{E}{e}}$$

$$= \sqrt{\frac{R^2}{R^2\cos^2\phi}}$$

$$= \frac{1}{\cos\phi} \tag{7.4.11}$$

From (7.4.1)

$$m_p = \cos\phi \tag{7.4.12}$$

Note that the four sets of distortion factors, (7.4.3) and (7.4.4), (7.4.6) and (7.4.7), (7.4.9) and (7.4.10), and (7.4.11) and (7.4.12) are all independent of longitude. Thus, in these cases, distortion in length if a function of latitude alone.

In order to find distortion in angles, substitute (7.4.1) into (7.2.13).

$$\sin(A-\alpha) = \frac{\left(m_p - \frac{1}{m_p}\right)}{\left(m_p + \frac{1}{m_p}\right)}\sin(A-\alpha)$$

$$\sin(A-\alpha) = \left(\frac{m_p^2 - 1}{m_p^2 + 1}\right)\sin(A+\alpha) \tag{7.4.13}$$

Equation (7.4.13) can now be solved for constant m_p by the iteration method of Section 7.2.

7.5 Distortion in Conformal Projections [20]

From Chapter 5, it is recalled that conformal projections are characterized by the fact that

$$m^2 = \frac{E}{e} = \frac{G}{g} \tag{7.5.1}$$

for an orthogonal system. Thus, from (7.1.3) and (7.1.4), at every point

$$m_p = m_m \tag{7.5.2}$$

This relationship makes the work involved in any derivation of linear distortions easier, since only one ratio of the first fundamental quantities needs to be evaluated. In this section, the polar stereographic, the Lambert conformal with one and two standard parallels, and the Mercator projection will be considered. These distortions will be based on the spheroid as the earth surface.

For the Lambert conformal projection with one standard parallel, using (7.1.4)

$$\begin{aligned} m &= m_p \\ &= m_m \\ &= \sqrt{\frac{G}{g}} \end{aligned} \tag{7.5.3}$$

Substitute (5.3.2) and (5.3.3), and apply the fundamental transformation matrix to (7.5.3).

$$\begin{aligned} m &= \sqrt{\frac{c_1^2\,\rho^2}{R_p^2\,\cos^2\phi}} \\ &= \frac{c_1\,\rho}{R_p\,\cos\phi} \end{aligned} \tag{7.5.4}$$

Substitute (5.3.14) and (5.3.21) into (7.5.4).

$$m = R_{p_0}\cos\phi_0 \times \left\{ \frac{\tan\left(\frac{\pi}{4} - \frac{\phi_1}{2}\right)\left(\frac{1 + e\sin\phi}{1 - e\sin\phi}\right)^{e/2}}{\tan\left(\frac{\pi}{4} - \frac{\phi_0}{2}\right)\left(\frac{1 + e\sin\phi_0}{1 - e\sin\phi_0}\right)^{e/2}} \right\}^{\sin\phi_0} \frac{1}{R_p\cos\phi} \tag{7.5.5}$$

For the two standard parallel case, the distortion in length is obtained in a manner similar to the one standard parallel case. From (7.5.4)

$$m = \frac{\rho \sin \phi_0}{R_p \cos \phi} \tag{7.5.6}$$

Equation (7.5.6) is then evaluated with the aid of (5.3.28), (5.3.31), and (5.3.32).

For the stereographic polar projection, the linear distortion along the parametric curves may be found by considering the plane and the spheroid. From the polar coordinates of the mapping plane

$$G = \rho^2 \tag{7.5.7}$$

For the spheroid, again

$$g = R_p^2 \cos^2 \phi \tag{7.5.8}$$

Substituting (7.5.7) and (7.5.8) into (7.5.1),

$$m = \frac{\rho}{R_p \cos \phi} \tag{7.5.9}$$

Substitute (5.4.6) into (7.5.9)

$$m = \frac{2a}{\sqrt{1 - e^2}} \left(\frac{1 - e}{1 + e}\right)^{e/2} \tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right)$$

$$\times \left(\frac{1 + e \sin \phi}{1 - e \sin \phi}\right)^{e/2} \left(\frac{1}{R_p \cos \phi}\right) \tag{7.5.10}$$

As a final effort in this section, we will consider the regular Mercator projection. Again, the spheroid has the first fundamental quantity

$$g = R_p^2 \cos^2 \phi \tag{7.5.11}$$

For the plotting surface

$$G = a^2 \tag{7.5.12}$$

Substitute (7.5.11) and (7.5.12) into (7.5.1)

$$m = \frac{a}{R_p \cos \phi} \tag{7.5.13}$$

Consider the linear distortions given in (7.5.5), (7.5.6), (7.5.10), and (7.5.13). Again, All of these equations depend only on latitude. In any practical evaluation of these distortions, it is sufficient to treat the earth as a sphere, rather than as a spheroid. This is done by letting $e = 0$ in the equations for the distortions. These equations then reduce to easily manageable forms.

For the distortion in angle at a point, substitute (7.5.2) into (7.2.13)

$$\sin(A - \alpha) = \frac{(m_p - m_p)}{(m_p + m_p)} \sin(A + \alpha)$$

$$= 0$$

$$A - \alpha = 0$$

$$A = \alpha \qquad (7.5.14)$$

Thus, one of the properties of the conformal projection is that angles are preserved in the transformation.

7.6 Distortions in Conventional Projections [1], [20]

Three of the most important conventional projections will be considered. These are the gnomonic, the azimuthal equidistant, and the polyconic. In these projections, $m_p \neq m_m$, and $m_p \neq 1/m_m$. Since there exists no simple relation between the two distortions in length along the parametric curves, it is necessary to solve for both. However, basic definitions make this simple in many cases.

Consider first the polar gnomonic projection. From the geometry of the case (Figure 6.1.1)

$$m_m = \frac{R \tan \delta \, d\lambda}{R \sin \delta \, d\lambda}$$

$$= \frac{1}{\cos \delta}$$

$$= \frac{1}{\sin \phi} \qquad (7.6.1)$$

For the distortion perpendicular to the meridian

$$m_p = \frac{dS}{ds}$$

$$= \frac{1}{\sin^2 \phi} \qquad (7.6.2)$$

For the azimuthal equidistant polar projection, the distance along the meridians is true, by definition. Thus,

$$m_m = 1 \tag{7.6.3}$$

The distance along the parallels on the map, as compared to those on the sphere follows from the geometry of a circular segment,

$$m_p = \frac{2\pi_a\left(\frac{\pi}{2} - \phi\right)}{2\pi a \cos\phi}$$

$$= \frac{\pi/2 - \phi}{\cos\phi} \tag{7.6.4}$$

The last projection to be considered is the regular polyconic. By the assumptions included in the derivation of the projection

$$m_p = 1 \tag{7.6.5}$$

The distortion along the meridians is, without proof, given by

$$m_m = 1 + \frac{\lambda^2}{2}\cos^2\lambda \quad [1] \tag{7.6.6}$$

For the gnomonic and azimuthal equidistant projections, the linear distortion is independent of longitude. However, the distortion in the meridian plane for the polyconic is a function of both latitude and longitude.

Distortion in angles at a point must be found by the numerical technique of (7.2.17).

7.7 Qualitative Comparisons [8], [22]

While a numerical approach is certainly useful qualitative comparisons of the various projections are also useful. This section will compare selected azimuthal, world, cylindrical, and conical projections.

Figure 7.7.1 compares five of the azimuthal polar projections: the equal area, the equidistant, the orthographic, the stereographic, and the gnomonic. Beginning with the orthographic, there is a steady gradation of parallel spacing, ending with the gnomonic. The orthographic projection suffers distortion as the equator is approached. The parallels are unequally spaced, and are bunched together close to the equator. The convergence of the parallels is not as severe for the equal area projection. The equidistant projection has equally spaced parallels. The stereographic and gnomonic projections have a divergence of the concentric parallels as the equator is approached. The distortion in the gnomonic projection is more severe, and the equator itself can never be portrayed.

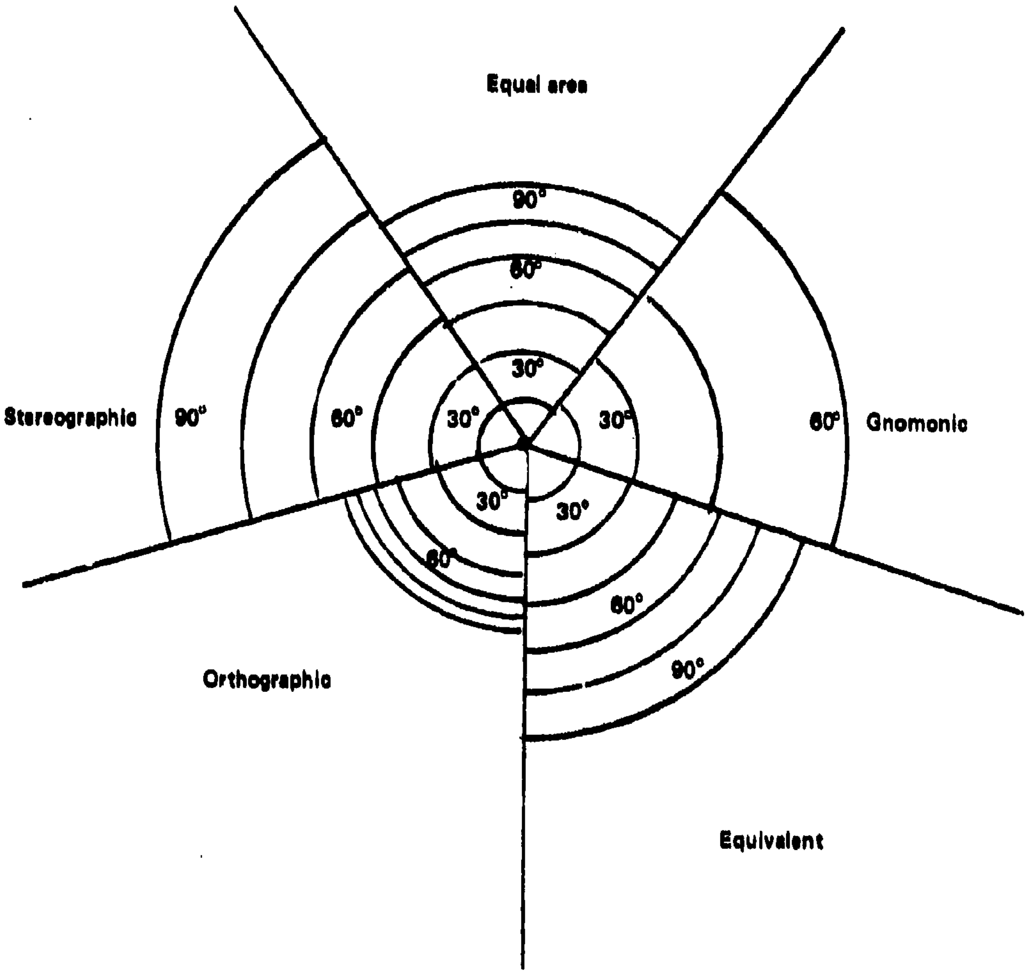


Figure 7.7.1. Comparison of azimuthal projections

In Figure 7.7.2, one quadrant of three equal area world maps of the same scale is plotted. The sinusoidal projection results in a pointed figure. However, the Mollweide is a smooth curve. The parabolic projection is in between the two in terms of distortion. Note also, there is a variation in the spacing of the parallels.

In Figure 7.7.3 is a comparison of the Mercator, the Plate Carée, and the cylindrical equal area. The Plate Carée has equal spacing along the meridian. The spacing for the Mercator increases as higher latitudes are reached. This is reversed in the equal area cylindrical. Here, the spacing decreases at higher latitudes.

Three conical projections of one standard parallel are compared in Figure 7.7.4. These are the equal area, the perspective, and the Lambert conformal. In the equal area projection, the parallels are closer at higher latitudes, and farther at lower latitude. In the Lambert conformal, the spacing diverges north and south of the standard parallel, but in a gradual way. In the perspective projection, the divergence is far more severe.

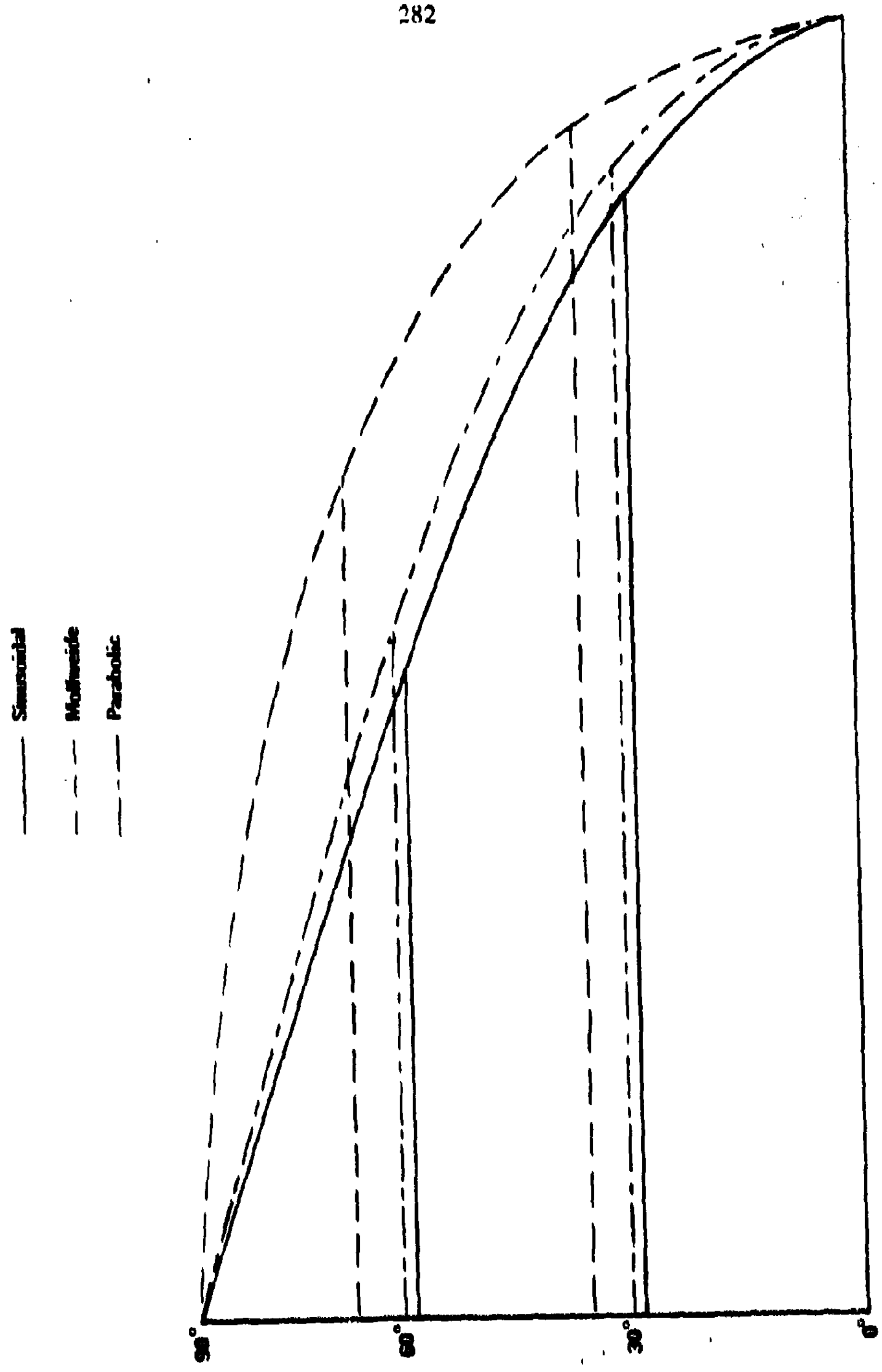


Figure 7.7.2. Comparison of equal area world maps

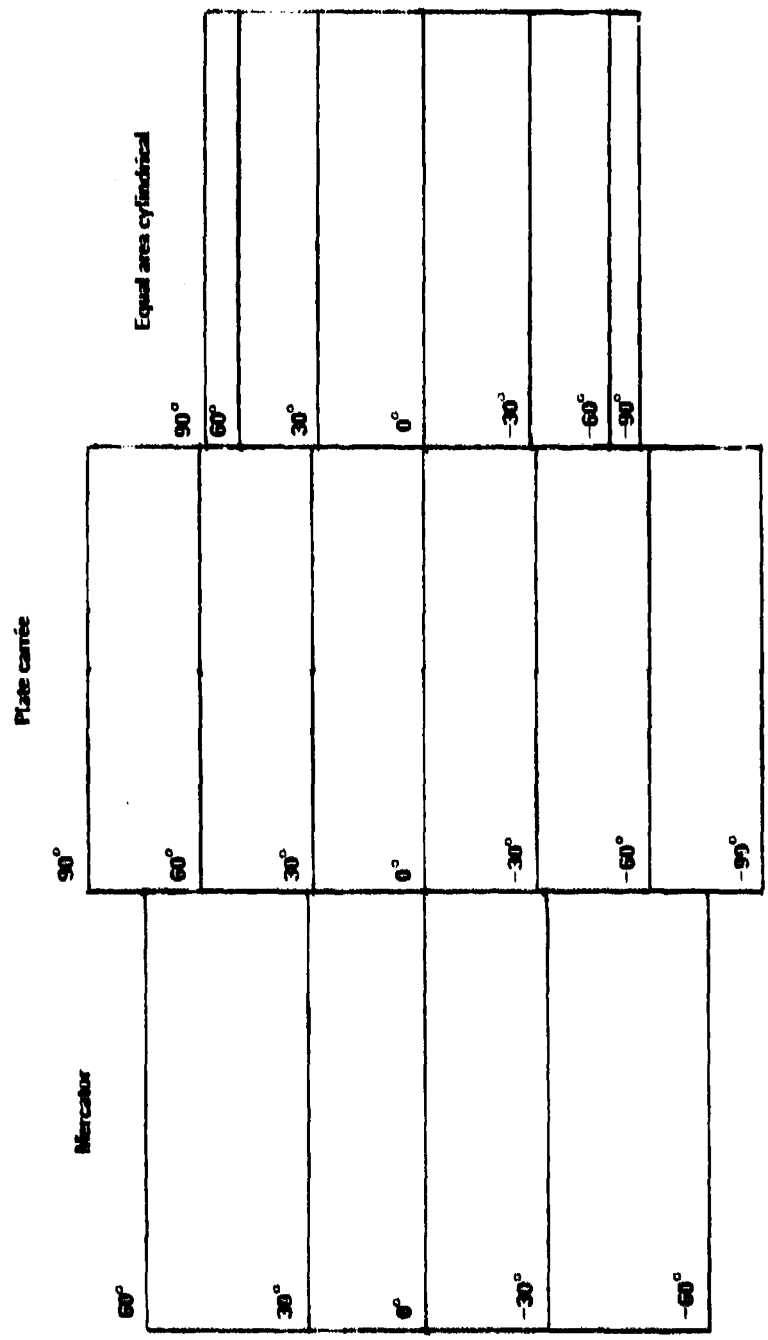


Figure 7.7.3. Comparison of cylindrical projections

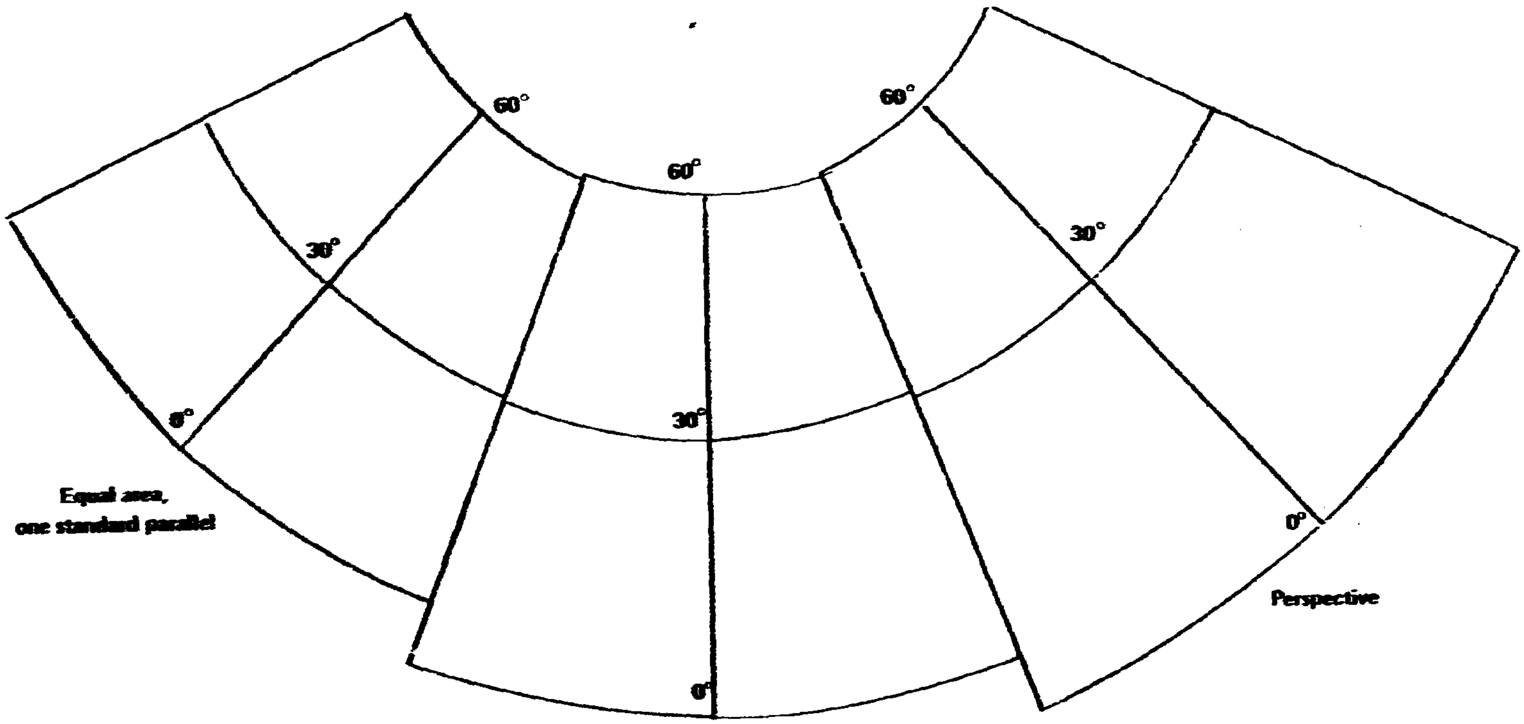


Figure 7.7.4. Comparison of conical projections

Chapter 8

CONCLUSION

This report has attempted to bring together in one document a fairly complete set of map projection equations. To this end, it was necessary to define the problem, and to introduce the terminology of mapping. Then, the basic transformations of mapping were derived. The model of the earth, with the current values of its parameters were then needed. Next, the three classes of map projections: equal area, conformal, and conventional, were considered one by one. For each class, a number of projection schemes were investigated. Finally, a means of numerically evaluating distortion was given. This has been an introduction to the theory of map projections.

However, of far more utility to the users of maps is the practice of map projections. Thus, in almost all cases, the final result is expressed in planar cartesian mapping coordinates. The formulas for the most useful of these projections were embedded in a computer program, and a variety of plotting tables were generated.

The level of mathematics chosen for the derivations was consistent with the strict intention of obtaining usable cartesian plotting equations. Previous publications have suffered from one or the other of two opposite extremes. Many present the projection with little or no derivation, and intend to follow mainly a graphical approach. In these, the spheroid is not considered. The opposite approach is one of mathematical elegance. The same derivation is followed in a number of ways that seem to thrive on complexity. In this report, the concepts of differential geometry were imposed as the unifying principles. In most cases, the derivations were relatively clean and straight-forward. In some other cases, such as the Parabolic projection, it was necessary to use a unique method. Still, the derivations remained utilitarian.

The plotting tables serve in a number of ways. First, they were used to provide the data points for the graticules which appear as figures in Chapters 4, 5, and 6. Second, those plotting tables which are general, and depend on no standard parallel, can be used to produce a usable grid for any application. The user can choose his central meridian, and plot from there. Since scale depends on a multiplicative factor, a user can apply an engineer's scale to enlarge or contract the grid to his own specifications. Finally, the plotting tables, both those general and particular, can serve a user as a check if he prepares a mapping computer program.

The computer program itself, which generated the plotting tables, is modularized. First of all, it can be used as a stand alone program to produce data points. However, any user can discard the calling program MAP, and adapt, with his own linkages, the subroutines POINT, GRID, AUTHAL, CONFRM, and the projection routines. Or, he can go one step farther, and use any or all of the projection subroutines by themselves in his own program. Thus, the program can be useful, in the future, to those who map by hand, use a digital/analog plotter, or use a CRT for computer graphics.

The chapter on distortion gives a numerical means of estimating the amount of distortion in a particular area of a map. Again, the methods of differential geometry gave a unified approach to this in most cases. In the remaining cases, it was necessary to use a brute force approach to compare the length on the map to the length on a sphere or spheroid.

Associated with the theory and practice of map projections is the philosophy of using map projections correctly. The rule, of course, is to choose a system that minimizes distortion to an acceptable level in the region of interest.

The precept for choosing the correct map for the required job is helped by the apparent natural adaptability of types of projections to certain areas of the earth. By consulting the projection tables and the figures, it is seen, that the azimuthal polar projections can easily, and with minimum distortion, handle areas immediately adjacent to the earth's poles. Likewise, the cylindrical projections are natural for the regions above and below the equator. The areas at mid-latitudes are conveniently spanned by conical projections of one or two standard parallels. Recall that in these projections, distortion is not a function of longitude. Thus, for a conformal representation of the world, a natural set is the Mercator between $\pm 30^\circ$ latitude, a Lambert conformal from $\pm 30^\circ$ to $\pm 60^\circ$, and the polar stereographic from $\pm 60^\circ$ to $\pm 90^\circ$. For an equal area approach, the same regions can be covered by an equal area cylindrical, an Albers, and the Lambert azimuthal, respectively.

If there is a desire to include an entire sphere or a hemisphere on a map, the polyconic, or the world statistical maps are the best approach. However, one must learn to live with the excessive distortion, or use interrupted variations. Note that only equal area and conventional projections can be used for a map of the entire earth. Extreme distortions at the periphery of conformal projections will not permit this.

Specific applications have called for the utilization of specific projections. Air and sea navigation have required the Mercator with the loxodrome, and the gnomonic, with the great circle over extended distances. Intermediate distances, such as a number of states or less, has fallen in the province of the polyconic and the Lambert conformal. In both of these projections, excessive distortion is not evident. Surveying systems have made use of the Transverse Mercator and Lambert conformal on large scale maps. Here, the curvature of the meridians and parallels are not noticeable. If one needs true distance and azimuths from a fixed point to anywhere in the world, the oblique azimuthal equidistant can accomodate. For large scale maps, and relatively short distances, any of the conformal maps, and also Bonne, can be used for relatively good approximations of distance and azimuth.

From the plotting tables, and the figures, it is apparent that at large scale, that is, over small areas, and in regions of minimal distortion, all of the projections approach the squares and rectangles of the Plate Carrée or the Carte Parallelogrammatique, except where meridian convergence is excessive. This is the reason that the grid system is useful on large scale maps.

Besides the use of natural and specialized projections, a number of variations are available by rotating the azimuthal plane, or the equatorial cylinder to cover areas of specific interest. Again, the plotting tables, and the subsequent figures indicate there are limited areas in each of these where distortion is minimized.

Thus, map projections provide a variety of methods of transforming from an undulating earth to a flat piece of paper, and obtaining, in the end, a fairly reliable representation.

BIBLIOGRAPHY

1. Adams, O. S., *General Theory of Polyconic Projections*, Special Publication 57, U. S. Coast and Geodetic Survey, 1934
2. Adams, O. S., *General Theory of Equivalent Projections*, Special Publication 236, U. S. Coast and Geodetic Survey, 1945
3. Adler, C. F., *Modern Geometry*, McGraw-Hill, 1958
4. Baker, R. M. L., and Makemson, M. W., *An Introduction to Astrodynamics*, Academic Press, 1960
5. Bomford, G., *Geodesy*, Oxford, 1962
6. Churchill, R. V., Brown, J. W., Verhey, R. F., *Complex Variables and Applications*, McGraw-Hill, 1960
7. Davies, R. E., Foote, F. S., Kelly, J. E., *Surveying: Theory and Practice*, McGraw-Hill, 1966
8. Deetz, C. H., and Adams, O. S., *Elements of Map Projection*, Special Publication 68, U. S. Coast and Geodetic Survey, 1944
9. Deetz, C. H., *Cartography, a Review and Guide*, Special Publication 205, U. S. Coast and Geodetic Survey, 1962
10. Goetz, A., *Introduction to Differential Geometry*, Addison-Wesley, 1970
11. Grant, H. E., *Practical Descriptive Geometry*, McGray-Hill, 1965
12. Hershey, A. V., *The Plotting of Maps on a CRT Printer*, NWL Report No. 1844, June 1963, U. S. Naval Weapons Laboratory
13. Hershey, A. V., *FORTRAN IV Programming for Cartography and Typography*, NWL Technical Report TR-2339, September 1969, U. S. Naval Weapons Laboratory
14. Hildebrand, F. B., *Introduction to Numerical Analysis*, McGraw-Hill, 1956
15. McBryde, F. W., and Thomas, P. D., *Equal Area Projections for World Statistical Maps*, Special Publication 245, U. S. Coast and Geodetic Survey, 1949
16. Middlemiss, R. R., Marks, J. L., Smart, J. R., *Analytic Geometry*, McGraw-Hill, 1945
17. Minor, D. C., and Denney, F. C., *College Geometry*, Prentice-Hall, 1972
18. Moffitt, F. H., and Bouchard, H., *Surveying*, Intext, 1975
19. Murdock, D. C., *Linear Algebra*, Wiley, 1970
20. Richardus, P., and Adler, R. K., *Map Projections for Geodesists, Cartographers, and Geographers*, North Holland, 1972
21. Seppelin, T. O., "The Department of Defense World Geodetic System, 1972," *The Canadian Surveyor*, 28(5), 496–506, Dec. 1974
22. Steers, J. A., *An Introduction to the Study of Map Projections*, University of London, 1962
23. Thomas, P. D., *Conformal Projections in Geodesy and Cartography*, Special Publication 251, U. S. Coast and Geodetic Survey, 1952
24. Wagner, K. H., *Kartographische Netzentwurfe*, Bibliographisches Institut Leipzig, January, 1972
25. *Tables for a Polyconic Projection of Maps*, Special Publication 2, U. S. Coast and Geodetic Survey, 1946

Appendix A.1

PROGRAM MAP

Program MAP is a means of generating cartesian mapping coordinates from the geodetic latitude and longitude of the spheroid.

Two basic options are available: selected points or a complete grid. If the point option is selected, a series of cards will be read, and then the desired transformation will be made. This is done by subroutine POINT. If a grid is required, this grid, at regular intervals, is generated. This is accomplished by subroutine GRID. Thus, POINT and GRID are the control subroutines of the program.

Twenty of the most useful of the map projections have been included in this program. To facilitate their use, two auxiliary subroutines are needed. These are AUTHAL, and CONFRM. AUTHAL converts from geodetic to authalic latitude for equal area projections. CONFRM transforms from the spheroid to produce conformal latitude and longitude on the conformal sphere.

The following plotting subroutines are included in this program. A reference is given to the section in which each set of plotting equations is derived.

Subroutine	*Projection*	*Section*
EQARAZ	Equal area azimuthal	4.3
EARCN1	Equal area conical, one standard parallel	4.2
EARCN2	Equal area conical, two standard parallels	4.2
EQARCL	Equal area cylindrical	4.5
BONNE	Bonne	4.4
PARAB	Parabolic	4.8
SINUS	Sinusoidal	4.6
MOLLW	Mollweide	4.7
HAMMER	Hammer–Aitoff	4.9
MERC	Mercator	5.2
STEREO	Stereographic	5.4
LAMBR1	Lambert conformal, one standard parallel	5.3
LAMBR2	Lambert conformal, two standard parallels	5.3
GNOM	Gnomonic	6.1
AZEQUD	Azimuthal equidistant	6.2
ORTHO	Orthographic	6.3
SIMCN1	Simple conical, one standard parallel	6.4

Subroutine	*Projection*	*Section*
SIMCN2	Simple conical, two standard parallels	6.4
SIMCL	Simple cylindrical, Miller	6.6
POLY	Polyconic	6.5

Figure A.1.1 shows the flow chart for the program, and the logical decision points. The input requirements and options, and the output are outlined in the paragraphs that follow.

The input is accomplished by five, or more cards. The format of these cards is now detailed.

First card (Format 8A10)
TI(I=1,8) Title of the case

Second card (Format 2F10.4)

A	Semi-major axis of the spheroid (meters)
E	Eccentricity of the spheroid

Third card (Format 5I5)

NCASE=0	Cases to follow
=1	Last case
NPROJ=1	Equal area azimuthal
=2	Equal area conical, one standard parallel
=3	Equal area conical, two standard parallels
=4	Equal area cylindrical
=5	Bonne
=6	Parabolic
=7	Sinusoidal
=8	Mollweide
=9	Hammer-Aitoff
=10	Mercator
=11	Stereographic
=12	Lambert conformal, one standard parallel
=13	Lambert conformal, two standard parallels
=14	Gnomonic
=15	Azimuthal equidistant
=16	Orthographic
=17	Simple conical, one standard parallel
=18	Simple conical, two standard parallels
=19	Simple conical, Miller
NPROJ=20	Polyconic
NPNT=0	Generate points
=1	Generate grid
NCON=0	Do not use the conformal sphere
=1	Convert from the spheroid to the conformal sphere
NAUTH=0	Do not use the authalic sphere
=1	Convert from the spheroid to the authalic sphere

Fourth card (Format 4F9.4,E14.8)

PHIO	Latitude of the origin, the pole of the auxiliary system, or the parallel of tangency (degrees)
LAMO	Longitude of the origin, or the pole of the auxiliary system (degrees)
P1	Lower standard parallel (degrees)
P2	Higher standard parallel (degrees)
SC	Scale factor

Fifth (and following) cards for POINT

PHI	Latitude of point (degrees)
LAM	Longitude of point (degrees)
NCARD=0	Points to follow
=1	Last point

Fifth card for GRID

PHI	First latitude for grid (degrees)
LAM	First longitude for grid (degrees)
DPHI	Increment of latitude (degrees)
DLAM	Increment of longitude of grid (degrees)
NPHI	Number of latitude points
NLAM	Number of longitude points

The output of the program is the same for both POINT and GRID.

PHI	Latitude of point (degrees)
LAM	Longitude of point (degrees)
X	x-plotting coordinate (meters)
Y	y-plotting coordinate (meters)

The printout of the calling program, the control subroutines, the auxiliary routines, and the mapping routines are attached. The output of this program is the plotting tables of Chapters 4, 5, and 6.

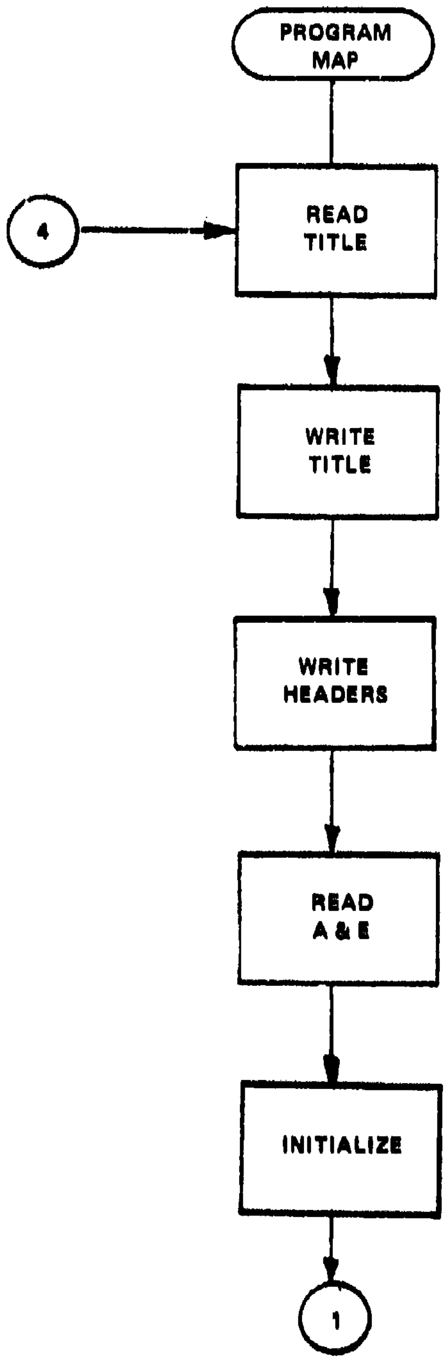


Figure A.1a. Flow chart

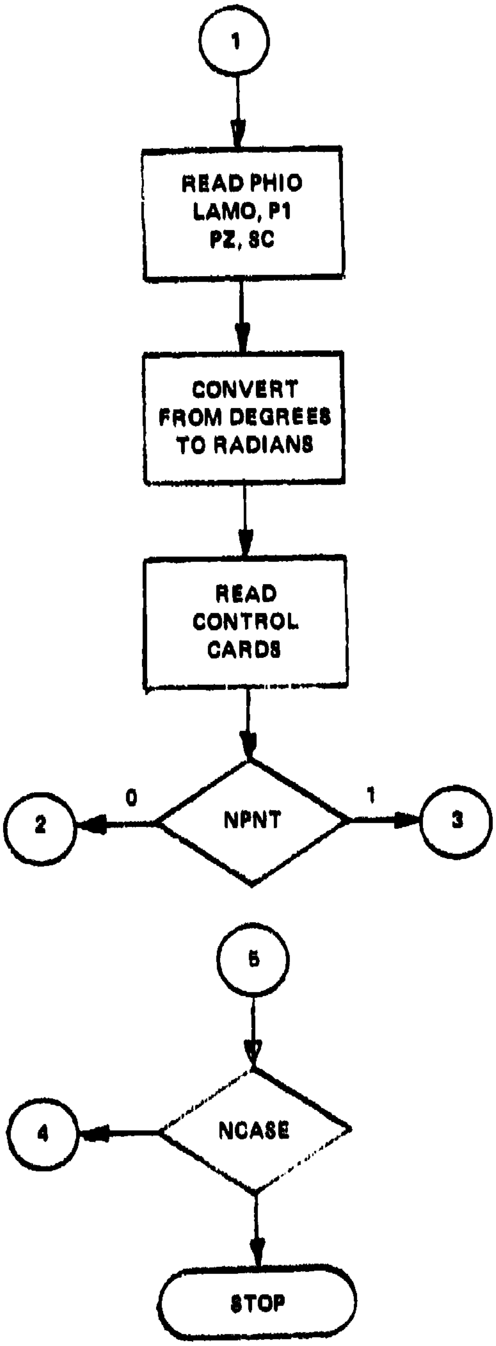


Figure A.1b. Flow chart

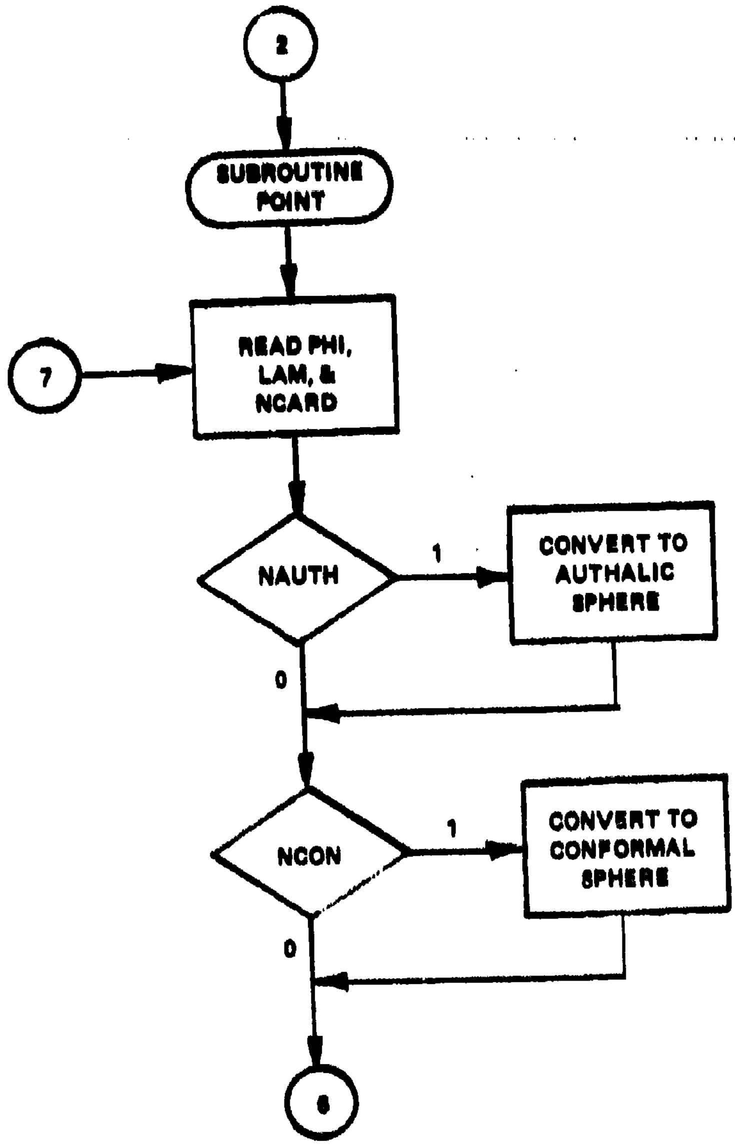


Figure A.1c. Flow chart

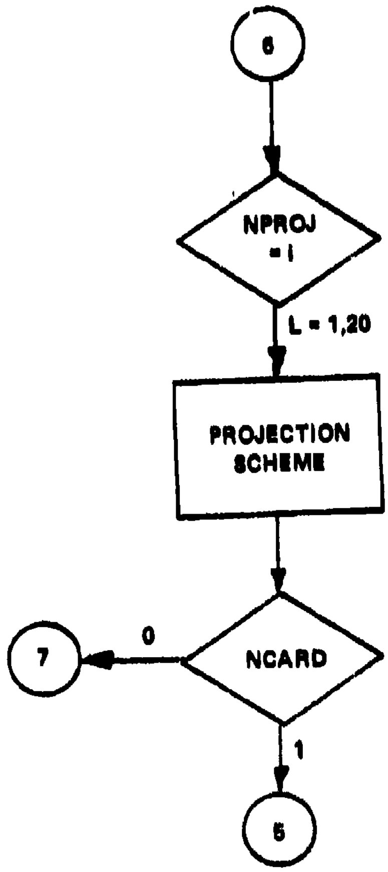


Figure A.1d. Flow chart

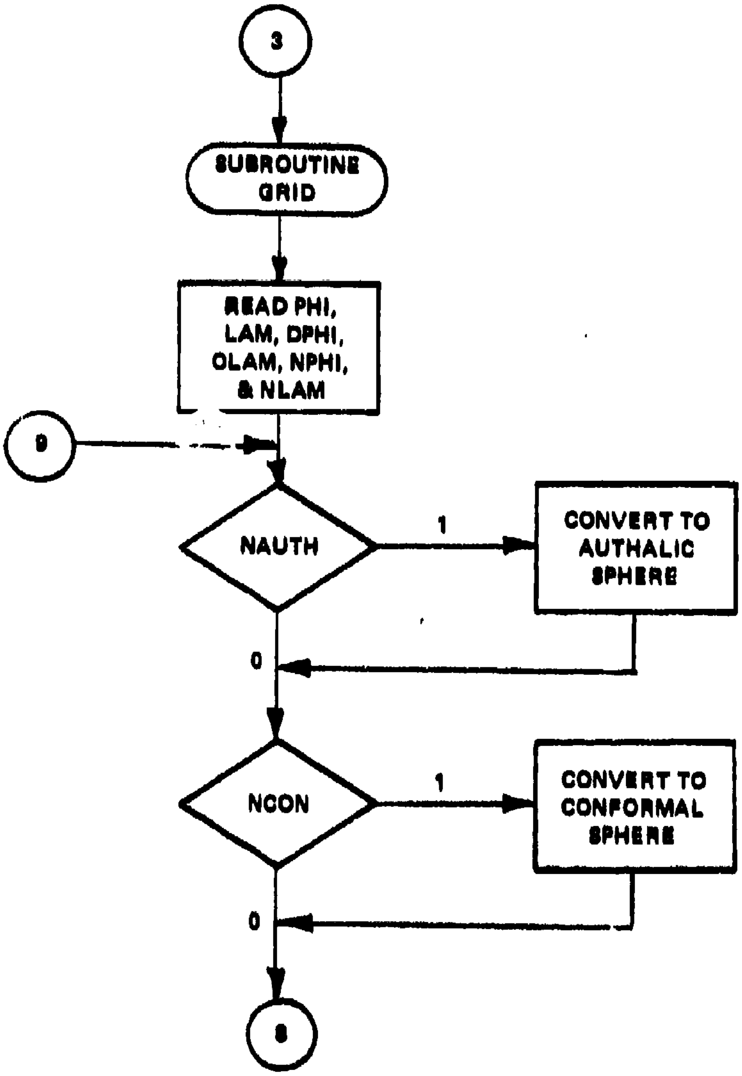


Figure A.1e. Flow chart

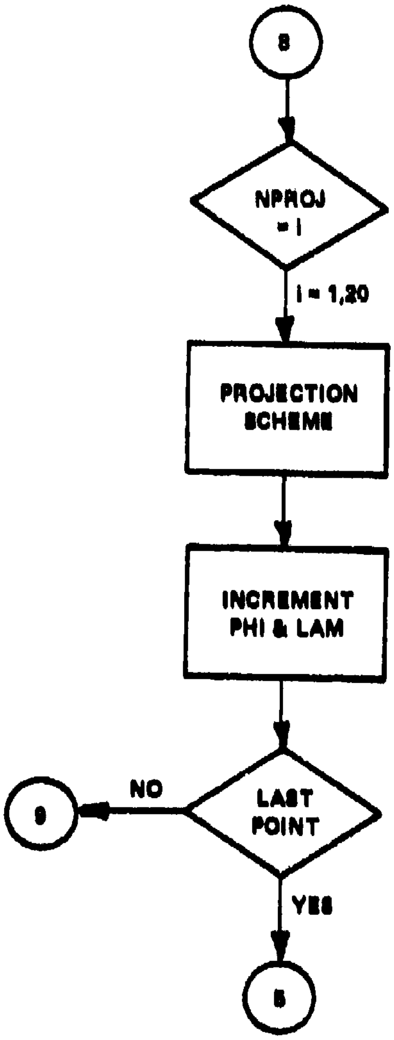


Figure A.1f. Flow chart

```
      PROGRAM MAP(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
      REAL LAMR,LAMOR,LAMO
      COMMON/CNST/A,E,S,PHIOR,LAMOR,R,RC,PI,PH1,PH2,RTD
      COMMON/INOUT/PHIR,LAMR,X,Y
      PI=3.14159
      RTD=57.295
    1 READ(5,2) T1,T2,T3,T4,T5,T6,T7,T8
    2 FORMAT(8A10)
      WRITE(6,3)
    3 FORMAT(1H1)
      WRITE(6,4) T1,T2,T3,T4,T5,T6,T7,T8
    4 FORMAT(5X,8A10)
      WRITE(6,102)
  102 FORMAT(6X,8HLATITUDE,5X,9HLONGITUDE,18X,1HX,18X,1HY)
      READ(5,7) A,E
    7 FORMAT(2F10.1)
      R=A
      RC=A
      PH1=0.
      PH2=0.
      LAMO=0.
      PHIO=0.
      READ(5,101) PHIO,LAMO,P1,P2,S
  101 FORMAT(4F9.4,E14.8)
      PHIOR=PHIO/RTD
      LAMOR=LAMO/RTD
      PH1=P1/RTD
      PH2=P2/RTD
      READ(5,5) NCASE,NPROJ,NPNT,NCON,NAUTH
    5 FORMAT(5I5)
      IF(NPNT.EQ.1) CALL GRID(NCON,NAUTH,NPROJ)
      IF(NPNT.EQ.1) GO TO 6
      CALL POINT(NCON,NAUTH,NPROJ)
    6 IF(NCASE.EQ.1) GO TO 1
      STOP
      END
```

```
      SUBROUTINE POINT(NCON,NAUTH,NPROJ)
      REAL LAM,LAMR,LAMO,LAMOR,LAMT,LAMOT
      COMMON/CNST/A,E,S,PHICR,LAMOR,R,RC,PI,PH1,PH2,RTD
      COMMON/INOUT/PHIR,LAMR,X,Y
  105 READ(5,104) PHI,LAM,NCARD
  104 FORMAT(2F9.4,I5)
      LAMR=LAM/RTD
      PHIR=PHI/RTD
      IF(NAUTH.EQ.1) CALL AUTHAL(PHIR,PHIT)
      IF(NCON.EQ.1) CALL CONFRM(PHIR,PHIT,LAMR,LAMT,LAMOT)
      IF(NAUTH.EQ.1.OR.NCON.EQ.1) PHIR=PHIT
      IF(NCON.EQ.1) LAMR=LAMT
      IF(NCON.EQ.1) LAMOR=LAMOT
      GO TO(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,
    1 CALL EQUARAZ                                        19,20) NPROJ
      GO TO 105
    2 CALL EARCN1
      GO TO 105
    3 CALL EARCN2
      GO TO 105
    4 CALL EQARCL
      GO TO 105
    5 CALL BONNE
      GO TO 105
    6 CALL PARAB
      GO TO 105
    7 CALL SINUS
      GO TO 105
    8 CALL MOLLW
      GO TO 105
    9 CALL HAMMER
      GO TO 105
   10 CALL MERC
      GO TO 105
   11 CALL STEREO
      GO TO 105
   12 CALL LAMBR1
      GO TO 105
   13 CALL LAMBR2
      GO TO 105
   14 CALL GNOM
      GO TO 105
   15 CALL AZEQUD
      GO TO 105
   16 CALL ORTHO
      GO TO 105
   17 CALL SIMCN1
      GO TO 105
   18 CALL SIMCN2
      GO TO 105
   19 CALL SIMCL
      GO TO 105
   20 CALL POLY
  105 WRITE(6,106) PHI,LAM,X,Y
  106 FORMAT(2(5X,F9.4),2(7X,F12.3))
```

```
      IF(NCARD.EQ.1) GO TO 103
      RETURN
      END
```

```
      SUBROUTINE GRID (NCON,NAUTH,NPROJ)
      REAL LAM,LAMR,LAMO,LAMOR,LAMT,LAMOT,LAMD
      COMMON/CNST/A,E,S,PHIOR,LAMOR,R,RC,PI,PH1,PH2,RTD
      COMMON/INOUT/PHIR,LAMR,X,Y
      READ(5,103) PHI,LAM,DPHI,DLAM,NPHI,NLAM
  103 FORMAT(4(F10.4),2I5)
      PHIR=PHI/RTD
      DPHIR=DPHI/RTD
      DLAMR=DLAM/RTD
      DO 104 I=1,NPHI
      LAMR=LAM/RTD
      DO 107 J=1,NLAM
      IF(NAUTH.EQ.1) CALL AUTHAL(PHIR,PHIT)
      IF(NCON.EQ.1) CALL CONFRM(PHIR,PHIT,LAMR,LAMT,LAMOT)
      IF(NAUTH.EQ.1.OR.NCON.EQ.1) PHIR=PHIT
      IF(NCON.EQ.1) LAMR=LAMT
      IF(NCON.EQ.1) LAMOR=LAMOT
      GO TO(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,
    1 CALL EQARAZ                                        19,20),NPROJ
      GO TO 105
    2 CALL EARCN1
      GO TO 105
    3 CALL EARCN2
      GO TO 105
    4 CALL EQARCL
      GO TO 105
    5 CALL BONNE
      GO TO 105
    6 CALL PARAB
      GO TO 105
    7 CALL SINUS
      GO TO 105
    8 CALL MCLLW
      GO TO 105
    9 CALL HAMMER
      GO TO 105
   10 CALL MERC
      GO TO 105
   11 CALL STEREO
      GO TO 105
   12 CALL LAMBR1
      GO TO 105
   13 CALL LAMBR2
      GO TO 105
   14 CALL GNOM
      GO TO 105
   15 CALL AZEQUD
      GO TO 105
   16 CALL ORTHO
      GO TO 105
   17 CALL SIMCN1
      GO TO 105
   18 CALL SIMCN2
      GO TO 105
   19 CALL SIMCL
```

```
      GO TO 105
   20 CALL POLY
  105 PHID=RTD*PHIR
      LAMD=RTD*LAMR
      WRITE(6,106) PHID,LAMD,X,Y
  106 FORMAT(2(5X,F9.4),2(7X,F12.3))
  107 LAMR=LAMR+DLAMR
  104 PHIR=PHIR+DPHIR
      RETURN
      END
```

```
      SUBROUTINE CONFRM(PHIR,PHIT,LAMR,LAMT,LAMCT)
      REAL LAMR,LAMOR,LAMT,LAMOT
      COMMON/CNST/A,E,S,PHIOR,LAMOR,F,RC,PI,PH1,PH2,RTD
      E2=E*E
      SNPO=SIN(PHIOR)
      SNPO2=SNPO*SNPO
      CSPO=COS(PHIOR)
      CSPO2=CSPO*CSPO
      CSPO4=CSPO2*CSPO2
      RPO=A/SQRT(1.-E2*SNPO2)
      RMO=A*(1.-E2)/(1-E2*SNPO2)**1.5
      C=SQRT(1.+E2*CSPO4/(1.-E2))
      LAMT=C*LAMR
      LAMCT=C*LAMOR
      SNP=SIN(PHIR)
      FAC1=((1.-E*SNP)/(1.+E*SNP))**(E/2.)
      FAC2=ALOG(PI/4.+PHIR/2.)
      TNPT=(FAC1*FAC2)**C
      PT=ATAN(TNPT)
      PHIT=2.*PT-PI/2.
      RETURN
      END
```

```
      SUBROUTINE AUTHAL(PHIR,PHIT)
      REAL LAMOR
      COMMON/CNST/A,E,S,PHIOR,LAMOR,R,RC,PI,PH1,PH2,RTD
      E2=E*E
      E4=E2*E2
      E6=E2*E4
      SNP=SIN(PHIR)
      SNP2=SNP*SNP
      SNP4=SNP2*SNP2
      SNP6=SNP2*SNP4
      FAC1=1.+.666667*E2*SNP2+.6*E4*SNP4+.571428*E6*SNP6
      FAC2=1.+.666667*E2+.6*E4+.571428*E6
      SNPT=SNP*FAC1/FAC2
      PHIT=ASIN(SNPT)
      R=A*SQRT((1-E2)*(1.+.666667*E2+.6*E4+.571428*E6))
      RETURN
      END
```

```
      SUBROUTINE EQARAZ
      REAL LAMR,LAMOR
      COMMON/CNST/A,E,S,PHIOR,LAMOR,R,RC,PI,PH1,PH2,RTD
      COMMON/INOUT/PHIR,LAMR,X,Y
      IF(PHIOR.LE.1.56207) GO TO 1
      FAC=SQRT(2.*(1.-SIN(PHIR)))
      X=R*S*FAC*COS(LAMR-LAMOR)
      Y=R*S*FAC*SIN(LAMR-LAMOR)
      RETURN
    1 SNP=SIN(PHIR)
      CSP=COS(PHIR)
      SNPO=SIN(PHIOR)
      CSPO=COS(PHIOR)
      SNL=SIN(LAMR-LAMOR)
      CSL=COS(LAMR-LAMOR)
      IF(PHIOR.LE..0001.AND.PHIR.LE..0001) ALF=PI/2.
      IF(PHIOR.LE..0001.AND.PHIR.LE..0001) GO TO 2
      ALF=ATAN2(SNL,(CSPO*SNP/CSP-SNPO*CSL))
    2 RHO=SQRT(2.*(1.-SNP*SNPO-CSP*CSPO*CSL))
      X=R*S*RHO*SIN(ALF)
      Y=R*S*RHO*COS(ALF)
      RETURN
      END
```

SUBROUTINE EARCN1 CDC 6600 FTN V3.0-P380 OPT=1

```
      SUBROUTINE EARCN1
      REAL LAMR,LAMOR
      COMMON/CNST/A,E,S,PHIOR,LAMCR,R,RC,PI,PH1,FH2,RTD
      COMMON/INOUT/PHIR,LAMR,X,Y
      SNPO=SIN(PHIOR)
      RHO=R*SQRT(1.+SNPO*SNFO-2.*SNPO*SIN(PHIR))/SNPC
      X=S*RHO*SIN((LAMR-LAMOR)*SNPO)
      Y=S*(R/TAN(PHIOR)-RHO*COS((LAMR-LAMOR)*SNPO))
      RETURN
      END
```

```
      SUBROUTINE EARCN2
      REAL LAMR,LAMOR
      COMMON/CNST/A,E,S,PHICR,LAMOR,R,RC,PI,PH1,PH2,RTD
      COMMON/INOUT/PHIR,LAMR,X,Y
      SN1=SIN(PH1)
      SN2=SIN(PH2)
      CS1=COS(PH1)
      CS2=COS(PH2)
      RHO1=2.*R*CS1/(SN1+SN2)
      RHO2=2.*R*CS2/(SN1+SN2)
      FAC1=.5*(RHO1+RHO2)
      FAC2=4.*R*R*(SN1-SIN(PHIR))/(SN1+SN2)
      FAC3=.5*(LAMR-LAMOR)*(SN1+SN2)
      FAC4=SQRT(RHO1*RHO1+FAC2)
      X=S*FAC4*SIN(FAC3)
      Y=S*(FAC1-FAC4*COS(FAC3))
      RETURN
      END
```

```
      SUBROUTINE EQARCL
      REAL LAMR,LAMOR
      COMMON/CNST/A,E,S,PHIOR,LAMOR,R,RC,PI,PH1,PH2,RTD
      COMMON/INOUT/PHIR,LAMR,X,Y
      X=R*S*(LAMR-LAMOR)
      Y=R*S*SIN(PHIR)
      RETURN
      END
```

```
      SUBROUTINE BONNE
      REAL LAMR,LAMOR
      COMMON/CNST/A,E,S,PHIOR,LAMOR,R,RC,PI,PH1,PH2,RTD
      COMMON/INOUT/PHIR,LAMR,X,Y
      RHOO=R*COS(PHIOR)/SIN(PHIOR)
      RHO=RHOO-R*(PHIR-PHIOR)
      X=RHO*S*SIN((LAMR-LAMOR)*COS(PHIR)*R/RHO)
      Y=S*(RHOO-RHO*COS((LAMR-LAMOR)*COS(PHIR)*R/RHO))
      RETURN
      END
```

```
      SUBROUTINE PARAB
      REAL LAMR,LAMOR
      COMMON/CNST/A,E,S,PHIOR,LAMOR,R,RC,PI,PH1,PH2,RTD
      COMMON/INOUT/PHIR,LAMR,X,Y
      Y=   R*S*SIN(PHIR/3.)*PI
      X=   (LAMR-LAMOR)*S*(2.*COS(.666667*PHIR)-1.)*R
      RETURN
      END
```

```
      SUBROUTINE SINUS
      REAL LAMR,LAMOR
      COMMON/CNST/A,E,S,PHIOR,LAMOR,R,RC,PI,PH1,PH2,RTD
      COMMON/INOUT/PHIR,LAMR,X,Y
      X=R*S*(LAMR-LAMOR)*COS(PHIR)
      Y=R*S*PHIR
      RETURN
      END
```

```
      SUBROUTINE MOLLW
      REAL LAMR,LAMOR
      COMMCN/CNST/A,E,S,PHIOR,LAMOR,R,RC,PI,PH1,PH2,RTD
      COMMON/INOUT/PHIR,LAMR,X,Y
      THT=PHIR
      SNF=SIN(PHIR)
      DO 1 I=1,50
      DTHT=(PI*SNF-2.*THT-SIN(2.*THT))/2./(1.+COS(2.*THT))
    1 THT=THT+DTHT
      Y=R*S*SIN(THT)*PI/2.
      X=    (LAMR-LAMOR)*R*S*COS(THT)
      IF(ABS(PHIR).GE.1.56207) X=0.
      RETURN
      END
```

```
      SUBROUTINE HAMMER
      REAL LAMR,LAMOR
      COMMON/CNST/A,E,S,PHIOR,LAMOR,R,RC,PI,PH1,PH2,RTD
      COMMON/INOUT/PHIR,LAMR,X,Y
      IF(ABS(TAN(PHIR)).LE..0001) ALF=PI/2.
      IF(ABS(TAN(PHIR)).LE..0001) GO TO 1
      ALF=ATAN2(SIN((LAMR-LAMOR)/2.),TAN(PHIR))
    1 RHO=SQRT(2.*(1.-COS(PHIR)*COS((LAMR-LAMOR)/2.)))
      X=2.*R*S*RHO*SIN(ALF)
      Y=R*S*RHO*COS(ALF)
      RETURN
      END
```

```
      SUBROUTINE MERC
      REAL LAMR,LAMOR
      COMMON/CNST/A,E,S,PHIOR,LAMOR,R,RC,PI,PH1,PH2,RTD
      COMMON/INOUT/PHIR,LAMR,X,Y
      IF(PHIOR.LE.1.56207) GO TO 1
      X=A*S*(LAMR-LAMOR)
      SNP=SIN(PHIR)
      FAC1=((1.-E*SNP)/(1.+E*SNP))**(E/2.)
      FAC2=TAN(PI/4.+PHIR/2.)
      Y=A*S*ALOG(FAC1*FAC2)
      RETURN
    1 SNP=SIN(PHIR)
       CSP=COS(PHIR)
      CSPO=COS(PHIOR)
      SNPO=SIN(PHIOR)
      SNL=SIN(LAMR-LAMOR)
      CSL=COS(LAMR-LAMOR)
      IF(PHIOR.LE..0001) GO TO 2
      ALF=ATAN2(SNL,(SNPO*CSL-CSPO*SNP/CSP))
      X=A*S*ALF
      FAC=(1.+SNP*SNPO+CSP*CSPO*CSL)/(1.-SNP*SNPO-CSP*CSPO*CSL)
      Y=A*S*ALOG(FAC)/2.
      RETURN
    2 X=A*S*ATAN2(SNL, SNP/CSP)
      Y=-A*S/2.*ALOG((1.+CSP*CSL)/(1.-CSP*CSL))
      RETURN
      END
```

```
      SUBROUTINE LAMBR1
      REAL LAMR,LAMOR
      COMMON/CNST/A,E,S,PHIOR,LAMOR,R,RC,PI,PH1,PH2,RTD
      COMMON/INOUT/PHIR,LAMR,X,Y
      SNP=SIN(PHIR)
      SNPO=SIN(PHIOR)
      RPO=A/SQRT(1.-E*E*SNPO*SNPO)
      RHOO=RPO*COS(PHIOR)/SNPO
      THT=(LAMR-LAMOR)*SNPO
      FAC1=TAN(PI/4.-PHIR/2.)
      FAC2=TAN(PI/4.-PHIOR/2.)
      FAC3=((1.+E*SNP)/(1.-E*SNP))**(E/2.)
      FAC4=((1.+E*SNPO)/(1.-E*SNPO))**(E/2.)
      RHO=RHOO*(FAC1*FAC3/FAC2/FAC4)**SNPO
      X=S*(RHO*SIN(THT))
      Y=S*(RHOO-RHO*COS(THT))
      RETURN
      END
```

SUBROUTINE LAMBR2 CDC 6600 FTN V3.0-P380 OPT=1 1

```
      SUBROUTINE LAMBR2
      REAL LAMR,LAMOR
      COMMON/CNST/A,E,S,PHIOR,LAMOR,R,RC,PI,PH1,PH2,RTD
      COMMON/INOUT/PHIR,LAMR,X,Y
      SNP=SIN(PHIR)
      SNP1=SIN(PH1)
      SNP2=SIN(PH2)
      CSP1=COS(PH1)
      CSP2=COS(PH2)
      RP1=A/SQRT(1.-E*E*SNP1*SNP1)
      RP2=A/SQRT(1.-E*E*SNP2*SNP2)
      FAC1=TAN(PI/4.-PH1/2.)
      FAC2=TAN(PI/4.-PH2/2.)
      FAC3=((1.+E*SNP1)/(1.-E*SNP1))**(E/2.)
      FAC4=((1.+E*SNP2)/(1.-E*SNP2))**(E/2.)
      FAC5=ALOG(RP1*CSP1/RP2/CSP2)
      SNPO=FAC5/ALOG(FAC1*FAC3/FAC2/FAC4)
      PSI=RP1*CSP1/(SNPO*FAC1*FAC3)
      THT=(LAMR-LAMOR)*SNPO
      RHO=PSI*(TAN(PI/4.-PHIR/2.)*((1.+E*SNP)/(1.-E*SNP))
      RH1=RP1*CSP1/SNPO                                  **(E/2.))
      X=S*RHO*SIN(THT)
      Y=S*(RH1-RHO*COS(THT))
      RETURN
      END
```

```
      SUBROUTINE STEREO
      REAL LAMR,LAMOR
      COMMON/CNST/A,E,S,PHIOR,LAMOR,R,RC,PI,PH1,PH2,RTD
      COMMON/INOUT/PHIR,LAMR,X,Y
      SNP=SIN(PHIR)
      IF(PHIOR.LE.1.56207) GO TO 1
      FAC1=((1.-E)/(1.+E))**(E/2.)
      FAC2=TAN(PI/4.-PHIR/2.)
      FAC3=((1.+E*SNP)/(1.-E*SNP))**(E/2.)
      RHO=2.*A*FAC1*FAC2*FAC3/SQRT(1.-E*E)
      THT=LAMR-LAMOR
      X=RHO*COS(THT)*S
      Y=RHO*SIN(THT)*S
      RETURN
    1 CSP=COS(PHIR)
      SNPO=SIN(PHIOR)
      CSPO=COS(PHIOR)
      CSL=COS(LAMR-LAMOR)
      SNL=SIN(LAMR-LAMOR)
      FAC=1.+SNPO*SNP+CSPO*CSP*CSL
      X=2.*A*CSP*SNL/FAC*S
      Y=2.*A*(CSPO*SNP-SNPO*CSP*CSL)/FAC*S
      RETURN
      END
```

```
      SUBROUTINE GNOM
      REAL LAMR,LAMOR
      COMMON/CNST/A,E,S,PHIOR,LAMOR,R,RC,PI,PH1,PH2,RTD
      COMMON/INOUT/PHIR,LAMR,X,Y
      SNP=SIN(PHIR)
      SNPO=SIN(PHIOR)
      CSP=COS(PHIR)
      CSPO=COS(PHIOR)
      SNL=SIN(LAMR-LAMOR)
      CSL=COS(LAMR-LAMOR)
      X=A*S*CSP*SNL/(SNPO*SNP+CSPO*CSP*CSL)
      Y=A*S*(CSPO*SNP-SNPO*CSP*CSL)/(SNPO*SNP+CSFO*CSP*CSL)
      RETURN
      END
```

```
SUBROUTINE AZEQUD
REAL LAMR,LAMOR
COMMON/CNST/A,E,S,PHIOR,LAMOR,R,RC,PI,PH1,PH2,RTD
COMMON/INOUT/PHIR,LAMR,X,Y
SNP=SIN(PHIR)
SNPO=SIN(PHIOR)
CSP=COS(PHIR)
CSPO=COS(PHIOR)
SNL=SIN(LAMR-LAMOR)
CSL=COS(LAMR-LAMOR)
PSI=ACOS(SNPO*SNP+CSPO*CSP*CSL)
CST=SNL*CSP/SIN(PSI)
SNT=(CSPO*SNP-SNPO*CSP*CSL)/SIN(PSI)
X=A*S*PSI*CST
Y=A*S*PSI*SNT
RETURN
END
```

```
SUBROUTINE  ORTHO           CDC 6600 FTN V3.0-P380 OPT=1  1

                 SUBROUTINE ORTHO
                 REAL LAMR,LAMOR
                 COMMON/CNST/A,E,S,PHIOR,LAMOR,R,RC,PI,PH1,PH2,RTD
                 COMMON/INOUT/PHIR,LAMR,X,Y
                 SNP=SIN(PHIR)
                 CSP=COS(PHIR)
                 SNPO=SIN(PHIOR)
                 CSPO=COS(PHIOR)
                 SNL=SIN(LAMR-LAMOR)
                 CSL=COS(LAMR-LAMOR)
                 IF(ABS(PHIOR).LE..0001) GO TO 1
                 PSI=ACOS(SNPO*SNP+CSPO*CSP*CSL)
                 THT=ATAN2((CSPO*SNP-SNPO*CSP*CSL),(SNL*CSP))
                 SNS=SIN(PSI)
                 X=A*S*SNS*COS(THT)
                 Y=A*S*SNS*SIN(THT)
                 RETURN
               1 PSI=ACOS(CSP*CSL)
                 SNS=SIN(PSI)
                 IF(ABS(SNL).LE..0001) THT=PI/2.
                 IF(ABS(SNL).LE..0001) GO TO 2
                 THT=ATAN2(TAN(PHIR),SNL)
               2 X=A*S*SNS*COS(THT)
                 Y=A*S*SNS*SIN(THT)
                 RETURN
                 END
```

```
      SUBROUTINE SIMCN1
      REAL LAMR,LAMOR
      COMMON/CNST/A,E,S,PHIOR,LAMOR,R,RC,PI,PH1,PH2,RTD
      COMMCN/INOUT/PHIR,LAMR,X,Y
      CTPO=COS(PHIOR)/SIN(PHIOR)
      X=A*S*(CTPO-PHIR+PHIOR)*SIN((LAMR-LAMOR)*SIN(PHIOR))
      Y=S*A*(CTPO-(CTPO-PHIR+PHIOR)*COS((LAMR-LAMOR)
                                     *SIN(PHICR)))
      RETURN
      END
```

```
      SUBROUTINE SIMCN2
      REAL LAMR,LAMOR
      COMMON/CNST/A,E,S,PHIOR,LAMOR,R,RC,PI,PH1,PH2,RTD
      COMMON/INOUT/PHIR,LAMR,X,Y
      CSP1=COS(PH1)
      CSP2=COS(PH2)
      C1=(CSP1-CSP2)/(PH2-PH1)
      RH1=(PH2-PH1)/(1.-CSP2/CSP1)
      RHO=RH1-PHIR+PH1
      X=A*S*RHO*SIN((LAMR-LAMOR)*C1)
      Y=A*S*(RH1-RHO*COS((LAMR-LAMOR)*C1))
      RETURN
      END
```

```
      SUBROUTINE SIMCL
      REAL LAMR,LAMOR
      COMMON/CNST/A,E,S,PHIOR,LAMOR,R,RC,PI,PH1,PH2,RTD
      COMMON/INOUT/PHIR,LAMR,X,Y
      X=A*S*(LAMR-LAMOR)/PI*2.
      Y=A*S*PHIR
      RETURN
      END
```

```
      SUBROUTINE POLY
      REAL LAMR,LAMOR
      COMMON/CNST/A,E,S,PHIOR,LAMOR,R,RC,PI,PH1,PH2,RTD
      COMMON/INOUT/PHIR,LAMR,X,Y
      IF(PHIR.NE.0.) GO TO 1
      Y=0.
      X=A*S*(LAMR-LAMOR)
      RETURN
    1 CSP=COS(PHIR)
      SNP=SIN(PHIR)
      SNL=SIN((LAMR-LAMOR)*SNP)
      CSL=COS((LAMR-LAMOR)*SNP)
      X=A*S*CSP*SNL/SNP
      Y=A*S*(PHIR-PHIOR+CSP*(1.-CSL)/SNP)
      RETURN
      END
```

Appendix A.2

ANCILLARY PROGRAMS

Four ancillary programs were required to generate the tables of Chapter 3. These are GEOCEN, CURVA, MERID, and CIRCLE. These programs are included in this appendix. In all of the programs, DLPHI is the increment in latitude in radians, and DLPHID is the increment in degrees. I is the number of increments to be calculated and printed. All four of these programs have the constants of the WGS-72 spheroid included in them.

GEOCEN gives geocentric latitude as a function of geodetic latitude, and geodetic latitude as a function of geocentric latitude. CURVA calculates the radii of curvature in the meridianal plane, and perpendicular to the meridianal plane, as a function of the geodetic latitude. MERID produces the distance along the meridianal ellipse corresponding to 1′ of arc as a function of geodetic latitude. Finally, CIRCLE gives the distance along a circle of parallel corresponding to 1′ of arc as a function of geodetic latitude.

PROGRAM CUFVA CDC 6600 FTN V3.0-P

```
      PROGRAM CURVA(INPUT,OUTPUT)
      PRINT 1
    1 FORMAT(1H1)
      E=.08181
      A=6378165.
      PHI=0.
      PHID=0.
      DLPHI=.0872665
      DLPHID=5.
      E2=E*E
      FAC=1.-E2
      DO 2 I=1,19
      SN=SIN(PHI)
      SN2=SN*SN
      RP=A/SQRT(1.-E2*SN2)
      RM=RP*FAC/(1.-E2*SN2)
      PRINT 3,PHID,RP,RM
    3 FORMAT(5X,F6.2,5X,F9.0,5X,F9.0)
      PHI=PHI+DLPHI
    2 PHID=PHID+DLPHID
      STOP
      END
```

PROGRAM GEOCEN CDC 6600 FTN V3.0-P38

```
      PROGRAM GEOCEN(INPUT,OUTPUT)
      PHI=0.
      PHID=0.
      E=.08181
      DELPHI=.0872665
      DLPHD=5.
      FAC=SQRT(1.-E*E)
      PRINT 3
    3 FORMAT(1H1)
      DO 1 I=1,18
      TPP=FAC*TAN(PHI)
      TP=TAN(PHI)/FAC
      PPD=ATAN(TPP)*57.2958
      PD=ATAN(TP)*57.2958
      PRINT 2,PHID,PPD,PHID,PD
    2 FORMAT (2X,F6.2,5X,F8.4,5X,F6.2,5X,F8.4)
      PHID=PHID+DLPHD
    1 PHI=PHI+DELPHI
      STOP
      END
```

```
      PROGRAM MERID (INPUT,OUTPUT)
      PRINT 1
    1 FORMAT(1H1)
      E=.08181
      E2=E*E
      E4=E2*E2
      E6=E2*E4
      A=6378165.
      CF1=1.-E2/4.-3.*E4/32
      CF1=1.-E2/4.-3.*E4/64.-5.*E6/256.
      CF2=3.*E2/8.+3.*E4/32.+45.*E6/1024.
      CF3=15.*E4/256.+45*E6/1024.
      CF4=35.*E6/3072.
      PHI=0.
      PHID=0.
      DLPHI=.0872665
      DLPHID=5.
      DO 2 I=1,19
      PHI1=PHI-1.45444E-4
      PHI2=PHI+1.45444E-04
      SN2=2.*(COS(PHI2+PHI1)*SIN(PHI2-PHI1))
      SN4=2.*(COS(2.*(PHI2+PHI1))*SIN(2.*(PHI2-PHI1)))
      SN6=2.*(COS(3.*(PHI2+PHI1))*SIN(3.*(PHI2-PHI1)))
      D=A*(CF1*(PHI2-PHI1)-CF2*SN2+CF3*SN4-CF4*SN6)
      PRINT 3, PHID,D
    3 FORMAT(5X,F6.2,5X,F9.3)
      PHI=PHI+DLPHI
    2 PHID=PHID+DLPHID
      STOP
      END
```

PROGRAM CIRCLE CDC 6600 FTN V3.0-P384

```
      PROGRAM CIRCLE (INPUT,OUTPUT)
      E=.08181
      A=6378165.
      DLAM=2.90888E-4
      PHI=0.
      PHID=0.
      DLPHI=.0872665
      DLPHID=5.
      PRINT 1
    1 FORMAT(1H1)
      DO 2 I=1,19
      SNP=SIN(PHI)
      SN2=SNP*SNP
      D=A*DLAM*COS(PHI)/SQRT(1.-E*E*SN2)
      PRINT 3, PHID,D
    3 FORMAT(5X,F6.2,5X,F9.3)
      PHI=PHI+DLPHI
    2 PHID=PHID+DLPHID
      STOP
      END
```

DISTRIBUTION

Defense Documentation Center
Cameron Station
Alexandria, VA 22314 (12)

Defense Printing Service
Washington Navy Yard
Washington, DC 20374

Library of Congress
Washington, DC 20540
ATTN: Gift and Exchange Division (4)

Defense Mapping Agency
Washington, DC 20360

Defense Mapping Agency Topographic Center
Washington, DC 20315

Defense Mapping Agency Aerospace Center
St. Louis, MO 63118

Defense Mapping Agency Hydrographic Center
Washington, DC 20390

Engineering Topographic Laboratory
Ft. Belvoir, VA 22060

National Oceanic and Atmospheric Adm.
National Ocean Survey
Rockville, MD 20850

VPI & SU
ESM Dept (Dr. L. Merrovitch)
Blacksburg, VA 24060 (4)

Local:

DK-01	
DK-10	(20)
DK-50	
DN-80	(2)
DN-83 (Pearson)	(45)
DX-21	(2)
DX-222	(6)
DX-40	
DX-43 (Aids)	